6.3, 6.6

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April 20, 2020

1 6.3, p.181

1. If A is a set of formulas, then $Th(A) = \{\sigma \mid A \vdash \sigma\}$ is a theory.

   Proof. Let ϕ be a formula such that $Th(A) \vdash \phi$. We must show $A \vdash \phi$ so that $\phi \in Th(A)$.
   $Th(A) \vdash \phi$ means there are $\sigma_i \in Th(A)$, $1 \leq i \leq n$, such that $D = (\sigma_1, \ldots, \sigma_n, \phi)$ is a deduction. Since $A \vdash \sigma_1, \ldots, A \vdash \sigma_n$, we know that, for each i, there are $\sigma_{ij} \in A$, $1 \leq j \leq n$ such that $(\sigma_{i1}, \ldots, \sigma_{in}, \sigma_i)$ is a deduction in A. When we line up these deductions so that D is the very last deduction listed, we have a deduction of ϕ in A. Thus, $\phi \in Th(A)$, and $Th(A)$ is a theory. ■

2 6.6, p.192

1. The universal quantification over sets A in $\mathbb{N}$ makes mathematical induction a principle of higher-order logic (second order, to be precise).

2. Suppose θ and η are two sentences that assert their own refutability in PA. That is,
   $$PA \vdash \left[\theta \leftrightarrow \neg Thm_{PA}\left(\overline{\ulcorner\theta\urcorner}\right)\right]$$
   and
   $$PA \vdash \left[\eta \leftrightarrow \neg Thm_{PA}\left(\overline{\ulcorner\eta\urcorner}\right)\right].$$
   Then, $PA \vdash \theta \leftrightarrow \eta$.

Proof.

$$
\begin{array}{ll}
PA \vdash \theta & \\
PA \vdash \neg Thm_{PA}(\overline{\ulcorner \theta \urcorner}) & def. \\
PA \vdash \neg \theta & (D1) \\
PA \vdash Thm_{PA}(\overline{\ulcorner \theta \urcorner}) & (D1) \\
PA \vdash Thm_{PA}(\overline{\ulcorner Thm_{PA}(\overline{\ulcorner \theta \urcorner}) \urcorner}) & (D2) \\
PA \vdash Thm_{PA}(\overline{\ulcorner \neg \theta \vee \eta \urcorner}) & (3) \\
PA \vdash Thm_{PA}(\overline{\ulcorner \theta \rightarrow \eta \urcorner}) & \\
PA \vdash \left[Thm_{PA}(\overline{\ulcorner \theta \urcorner}) \wedge Thm_{PA}(\overline{\ulcorner \theta \rightarrow \eta \urcorner}) \right] & (4, 7) \\
PA \vdash Thm_{PA}(\overline{\ulcorner \eta \urcorner}) & (D3) \\
PA \vdash \eta & (D1)
\end{array}
$$

Thus, $PA \vdash \theta \rightarrow \eta$.

$$
\begin{array}{lll}
PA \vdash \eta & & (1) \\
PA \vdash \neg Thm_{PA}(\overline{\ulcorner \eta \urcorner}) & def. & (2) \\
PA \vdash \neg \eta & (D1) & (3) \\
PA \vdash Thm_{PA}(\overline{\ulcorner \eta \urcorner}) & (D1) & (4) \\
PA \vdash Thm_{PA}(\overline{\ulcorner Thm_{PA}(\overline{\ulcorner \eta \urcorner}) \urcorner}) & (D2) & (5) \\
PA \vdash Thm_{PA}(\overline{\ulcorner \neg \eta \vee \theta \urcorner}) & (13) & (6) \\
PA \vdash Thm_{PA}(\overline{\ulcorner \eta \rightarrow \theta \urcorner}) & & (7) \\
PA \vdash \left[Thm_{PA}(\overline{\ulcorner \eta \urcorner}) \wedge Thm_{PA}(\overline{\ulcorner \eta \rightarrow \theta \urcorner}) \right] & (14, 17) & (8) \\
PA \vdash Thm_{PA}(\overline{\ulcorner \theta \urcorner}) & (D3) & (9) \\
PA \vdash \theta & (D1) & (10)
\end{array}
$$

Therefore, $PA \vdash \theta \leftrightarrow \eta$. ■