# 3.2

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February 19, 2020

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1. If $\Sigma$ is inconsistent and $\phi$ is an $\mathcal{L}$-formula, then $\Sigma \vdash \phi$.

   *Proof.* $\Sigma$ is inconsistent
   $\Sigma \vdash \bot$
   No truth assignment makes $\bot_P$ true
   $\phi_P$ is true for every truth assignment that $\bot_P$ is true
   $\phi_P$ is a propositional consequence of $\{\bot\}_P$
   $(\{\bot\}, \phi)$ is a rule of inference of type (PC)
   $\therefore \Sigma \vdash \phi$ ∎

2. Let $\Sigma_0 \subseteq \Sigma_1 \subseteq \Sigma_2 \subseteq \cdots$ be such that each $\Sigma_i$ is a consistent set of sentences in a language $\mathcal{L}$. Then $\bigcup \Sigma_i$ is consistent.

   *Proof.* Deny
   $\bigcup \Sigma_i$ is inconsistent
   $\bigcup \vdash \bot$
   There is a deduction $D = (\Gamma, \bot)$, where $\Gamma \subseteq \bigcup \Sigma_i$
   $\Gamma$ is finite $\implies \Gamma \subset \Sigma_j$, for some $j$.
   $\Sigma_j, \cdots \vdash \bot$
   $\Sigma_j, \ldots$ are inconsistent, a contradiction ∎

7. The equivalence relation $\sim$ on the set $T$ of variable-free terms of the language $\mathcal{L}'$ defined by

   $$t_1 \sim t_2 \text{ if and only if } (t_1 = t_2) \in \Sigma'$$

   is an equivalence relation.

*Proof.* Note that $(t_1 = t_1) \in \Sigma'$ since this is the logical axiom E1. So, $\sim$ is reflexive.

Let $\Gamma = \{t_1 = t_2, t_2 = t_3\}$ and let $\phi :\equiv t_1 = t_3$. If $\Gamma_P = \{A, B\}$, then when $A \wedge B$ is true, $\phi_P$ is true. So, $\phi$ is a propositional consequence of $\Gamma$. Hence, $(\Gamma, \phi)$ is a rule of inference of type PC, and $\Sigma' \vdash \phi$.

Or $\Sigma' \vdash (t_1 = t_3)$ by Thm 2.7.1. ■