5.7-5.8

Andrew Lounsbury

April 1, 2020

1 5.7, p.142

1. (a)

$$\begin{aligned}
\ulcorner(\forall v_3)(v_3 + 0 = v_4)\urcorner &= \langle 5, \ulcorner v_3 \urcorner, \ulcorner v_3 + 0 = 4 \urcorner \rangle \\
&= \langle 5, \langle 6 \rangle , \langle 7, \ulcorner v_3 + 0 \urcorner, \ulcorner v_4 \urcorner \rangle \rangle \\
&= \left\langle 5, 2^7, \langle 7, \langle 13, \ulcorner v_3 \urcorner, \ulcorner 0 \urcorner \rangle , \langle 8 \rangle \rangle \right\rangle \\
&= \left\langle 5, 2^7, \left\langle 7, \langle 13, \langle 6 \rangle , \langle 9 \rangle \rangle , 2^9 \right\rangle \right\rangle \\
&= \left\langle 5, 2^7, \left\langle 7, \left\langle 13, 2^7, 2^{10} \right\rangle , 2^9 \right\rangle \right\rangle \\
&= 2^5 3^{2^7} 5^{\left[2^7 3^{\left(2^{13} 3^{2^7} 5^{2^{10}} \right)} 5^{2^9} \right]}
\end{aligned}$$

(b)

$$\begin{aligned}
\ulcorner SSSS0 \urcorner &= \langle 11, \ulcorner SSS0 \urcorner \rangle \\
&= \langle 11, \langle 11, \ulcorner SS0 \urcorner \rangle \rangle \\
&= \langle 11, \langle 11, \langle 11, \ulcorner S0 \urcorner \rangle \rangle \rangle \\
&= \langle 11, \langle 11, \langle 11, \langle 11, \langle 9 \rangle \rangle \rangle \rangle \rangle \\
&= \left\langle 11, \left\langle 11, \left\langle 11, \left\langle 11, 2^9 \right\rangle \right\rangle \right\rangle \right\rangle \\
&= 2^{11} 3^{\left[2^{11} 3^{\left[2^{11} 3^{\left[2^{11} 3^{2^9} \right]} \right]} \right]}
\end{aligned}$$

2. (a)

$$2^8 3^{\left[2^{14}3^{\left(2^{18}3^9 5^{1025}+1\right)}5^{\left(2^{18}3^{33}5^{1025}+1\right)}+1\right]}5^{\left[2^{18}3^{129}5^{1025}+1\right]}$$
$$\left\langle 7, 2^{14}3^{\left(2^{18}3^9 5^{1025}+1\right)}5^{\left(2^{18}3^{33}5^{1025}+1\right)}, 2^{18}3^{129}5^{1025}\right\rangle$$
$$\langle 7, \langle 13, 2^{18}3^9 5^{1025}, 2^{18}3^{33}5^{1025}\rangle, \langle 17, 128, 1024\rangle\rangle$$
$$\langle 7, \langle 13, \langle 17, 8, 1024\rangle, \langle 17, 32, 1024\rangle\rangle, \langle 17, 128, 1024\rangle\rangle$$
$$\langle 7, \langle 13, \langle 17, 2^3, 2^{10}\rangle, \langle 17, 2^5, 2^{10}\rangle\rangle, \langle 17, 2^7, 2^{10}\rangle\rangle$$
$$\langle 7, \langle 13, \langle 17, \langle 2\rangle, \langle 9\rangle\rangle, \langle 17, \langle 4\rangle, \langle 9\rangle\rangle\rangle, \langle 17, \langle 6\rangle, \langle 9\rangle\rangle\rangle$$
$$\langle 7, \langle 13, \langle 17, \ulcorner v_1 \urcorner, \ulcorner 0 \urcorner\rangle, \langle 17, \ulcorner v_2 \urcorner, \ulcorner 0 \urcorner\rangle\rangle, \langle 17, \ulcorner v_3 \urcorner, \ulcorner 0 \urcorner\rangle\rangle$$
$$\langle 7, \langle 13, \ulcorner Ev_1 0 \urcorner, \ulcorner Ev_2 0 \urcorner\rangle, \ulcorner Ev_3 0 \urcorner\rangle$$
$$\langle 7, \ulcorner +Ev_1 0 Ev_2 0 \urcorner, \ulcorner Ev_3 0 \urcorner\rangle$$
$$= +Ev_1 0 Ev_2 0 Ev_3 0$$

(b)

$$2^2 3^{\left[2^{20}3^{1025}5^{\left(2^{12}3^{1025}+1\right)}+1\right]}$$
$$\left\langle 1, 2^{20}3^{1025}5^{(2^{12}3^{1025}+1)}\right\rangle$$
$$\langle 1, \langle 19, 1024, 2^{12}3^{1025}\rangle\rangle$$
$$\langle 1, \langle 19, 1024, \langle 11, 1024\rangle\rangle\rangle$$
$$\langle 1, \langle 19, 1024, \ulcorner Sv_5 \urcorner\rangle\rangle$$
$$\langle 1, \ulcorner v_5 S v_5 \urcorner\rangle$$
$$\neg < v_5 S v_5$$

(c)

$$2^6 3^9 5^{\left(2^8 3^9 5^9+1\right)}$$
$$\langle 5, 8, 2^8 3^9 5^9\rangle$$
$$\langle 5, 8, \langle 7, 8, 8\rangle\rangle$$
$$\langle 5, 8, \langle 7, \langle 2\rangle, \langle 2\rangle\rangle\rangle$$
$$\langle 5, 8, \ulcorner = v_1 v_1 \urcorner\rangle$$
$$(\ \forall v_1)(= v_1 v_1)$$

2 5.8, p.147

1. (a) $\underline{Term(\phi)}$: not a formula, the argument for $Term$ must be a (Gödel) number, not an $\mathcal{L}_{NT}$-formula ϕ

   (b) $\underline{Term(\ulcorner\phi\urcorner)}$: this is a formula

   (c) $\underline{Term(\overline{\ulcorner\phi\urcorner})}$: not a formula, the argument for $Term$ must be a (Gödel) number, not an $\mathcal{L}_{NT}$-formula ϕ

$$\overline{\ulcorner\phi\urcorner} :\equiv \underbrace{S\cdots S}_{\ulcorner\phi\urcorner \text{ times}} 0$$

5. If a is a natural number greater than or equal to 1, then $p_a \leq 2^{a^a}$, where p_a is the a-th prime number.

   Proof. Base case: If $a = 1$, then, $p_a = p_1 = 2 \leq 2 = 2^{1^1} = 2^{a^a}$.
   Induction: Suppose $p_k \leq 2^{k^k}$ for some arbitrary natural number k. Note that $p_{k+1} \leq p_{\min}$, where $p_{\min}$ is the smallest prime factor of $p_1 \cdots p_k - 1$. Also, note that $p_1 \cdots p_k = 2p_2 \cdots p_k$ is even, so $p_1 \cdots p_k - 1$ is odd. Hence, $p_{k+1} < 3 \leq p_{\min}$. We know $p_{k+1} < 3$ because we would otherwise have $3 \leq p_{k+1} < p_{\min}$, contradicting $p_{k+1} \leq p_{\min}$. Then, $p_{k+1} < 3 < 2^{2^2} \leq 2^{(k+1)^{(k+1)}}$, as desired. ■