2.5 - 2.7

Andrew Lounsbury

February 16, 2020

1 2.5, p.58

2. (E1) and (E3) are valid.

   Proof. Let $\mathfrak{A}$ be an $\mathcal{L}$-structure and s an assignment function into $\mathfrak{A}$. It is obvious that $\overline{s}(x)$ is the same as $\overline{s}(x)$, so $\mathfrak{A} \models (x = x)[s]$ for any $\mathfrak{A}$ with any s. Hence, (E1) is valid.
   For (E3), suppose $\mathfrak{A} \models [(x_1 = y_1) \wedge (x_2 = y_2) \wedge \cdots \wedge (x_n = y_n)][s]$. That is, $s(x_i)$ is $s(y_i)$ for each i.

   $$\begin{aligned} (\overline{s}(x_1), \ldots, \overline{s}(x_n)) &= (s(x_1), \ldots, s(x_n)) \in A^n, \\ &\stackrel{?}{=} (x_1, \ldots, x_n) \in R^{\mathfrak{A}} \\ (\overline{s}(y_1), \ldots, \overline{s}(y_n)) &= (s(y_1), \ldots, s(y_n)) \in A^n. \\ &\stackrel{?}{=} (y_1, \ldots, y_n) \in R^{\mathfrak{A}}. \end{aligned}$$

   Hence, $\mathfrak{A} \models R(x_1, \ldots, x_n)[s]$ and $\mathfrak{A} \models R(y_1, \ldots, y_n)[s]$ for any $\mathfrak{A}$ and any s. Since $x_i = y_i$ for each i, we know that $\mathfrak{A} \models (R(x_1, \ldots, x_n) = R(y_1, \ldots, y_n))[s]$ for any $\mathfrak{A}$ and any s. ■

4. If x is not free in ψ, then $(\phi \to \psi) \models [(\exists x \phi) \to \psi]$.

   Proof. Suppose x is not free in ψ and suppose that $\mathfrak{A} \models (\phi \to \psi)$ for some structure $\mathfrak{A}$. So, if $\mathfrak{A} \models \phi[s_1]$ for any s_1, then $\mathfrak{A} \models \psi[s_2]$ for any s_2. Now, suppose $\mathfrak{A} \models (\exists x \phi)[s]$ so that there is an element a in A for which $\mathfrak{A} \models \phi[s(x|a)]$ for any s.
   We must show $\mathfrak{A} \models \psi[s]$ for any s. The antecedent tells us that $\mathfrak{A} \models$

$\psi[t(x|a)]$ for some element a in A and for any t. Since x is not free in ψ, any other variable is substitutable for x in ϕ. Hence, $\mathfrak{A} \models \psi[t]$ for any assignment function t. Thus, $\mathfrak{A} \models [(\exists x\phi) \to \psi]$, and $(\phi \to \psi) \models [(\exists x\phi) \to \psi]$. ■

2 2.7, p.66

4. Let η be a sentence. Then, $\Sigma \vdash \eta$ if and only if $\Sigma \cup (\neg\eta) \vdash [(\ \forall x)x = x] \wedge \neg[(\ \forall x)x = x]$.

   Proof. Suppose $\Sigma \vdash \eta$. From η, we can deduce

   $$[(\ \forall x)x = x] \vee \neg[(\ \forall x)x = x] \to \eta$$

   since $[(\ \forall x)x = x] \vee \neg[(\ \forall x)x = x]$ is a tautology. We then deduce the contrapositive $\neg\eta \to [(\ \forall x)x = x] \wedge \neg[(\ \forall x)x = x]$. So,

   $$\Sigma \vdash (\neg\eta \to [(\ \forall x)x = x] \wedge \neg[(\ \forall x)x = x]).$$

   By the Deduction Theorem, $\Sigma \cup (\neg\eta) \vdash [(\ \forall x)x = x] \wedge \neg[(\ \forall x)x = x]$. Now, suppose

   $$\Sigma \cup (\neg\eta) \vdash [(\ \forall x)x = x] \wedge \neg[(\ \forall x)x = x],$$

   so that

   $$\Sigma \vdash (\neg\eta \to [(\ \forall x)x = x] \wedge \neg[(\ \forall x)x = x])$$

   by the Deduction Theorem. The contrapositive of this results in a tautology implying η:

   $$\Sigma \vdash ([(\ \forall x)x = x] \vee \neg[(\ \forall x)x = x] \to \eta).$$

   Therefore, $\Sigma \vdash \eta$. ■

5. Let P be a unary relation symbol. Then, $\vdash [(\ \forall x)P(x)] \to [(\exists x)P(x)]$.

   Proof. Suppose $\vdash (\ \forall x)P(x)$ but $\neg(\exists x)P(x)$. So, for every x, $P(x)$ is true, and there is no x such that $P(x)$ is true. This is a blatant contradiction. ■