6.3-6.6

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1 6.3, p.174

proof of Lemma 6.3.5 The authors use Lemma 6.3.3 to assert the existence of the Σ-formula $AxiomOfA(x)$. In Lemma 6.3.3, the Σ-formula in question is

$$\phi(v_1) :\equiv \exists z \exists y \exists c(Num(v_1, z) \wedge Sub(\overline{k}, \overline{8}, z, y) \wedge Deduction(c, y)).$$

Are they saying that this is what $AxiomOfA(x)$ is?
HETZEL: Believe it or not, yes. But, it's not as strange as you may initially think. Keep in mind that the k is defined *in terms of* ψ, which represents AxiomOfA. So, *AxiomOfA* is still intimately tied to AxiomOfA.
Also, is the replacement of the Δ-formula $AxiomOfN(e)$ with $AxiomOfA(e)$ the only change that turns $Deduction_A(c, v_1)$ into a Σ-formula?
HETZEL: Indeed. Keep in mind that the definition of *AxiomOfA* involves unbounded existential quantification.

2 6.4, p.182

Thm 6.4.5 Given Proposition 6.4.3, doesn't Theorem 6.4.5 follow from Propositions 6.4.3 and 6.4.4? HETZEL: Actually, it works the other way, that is, Theorem 6.4.5 + Proposition 6.4.3 $\rightarrow$ Proposition 6.4.4. For suppose that a theory is ω-consistent, recursive, and extends N. Since the theory is ω-consistent, it is consistent by Proposition 6.4.3. However, Theorem 6.4.5 then guarantees that the theory is incomplete.

6.4.4: A is an ω-consistent and recursive set of axioms extending N
$\overset{6.4.3}{\Longrightarrow}$ A is a consistent and recursive set of axioms extending N
$\overset{\text{same as } 6.4.5}{\Longrightarrow}$ A is incomplete

3 6.5, p.185

p.185 Is this finite restriction of $\mathcal{L}_{NT}$ only used for "naming" the outline of the proof of Theorem 6.5.1? HETZEL: It is the only place I am aware of where this finite restriction of $\mathcal{L}_{NT}$ is used.

p.186 In $Q \vdash ((\forall x)(\phi(x) \leftrightarrow x = \overline{n}))$, why is there a universal quantifier when it just ends up saying that x has to be some specific natural number n?HETZEL: Because we need what follows the $\vdash$ symbol to be a sentence.
Do all formulas have to name a number? I'm tempted not to think so since the definition of "naming" seems to say (as in the last paragraph of the section) that a number named by a formula is the "number that makes the formula true." But surely there are formulas that are true or provable for not just one natural number.HETZEL: No, not all formulas have to name a number. Your intuition is right on track.

Is "naming" used for anything other this one outline of proof? They really make it stand out but seem to use it for just this one thing. HETZEL: I am not aware of "naming" being used for anything else.

4 6.6, p.187

p.191 At the end of the paragraph preceding Corollary 6.6.4, the authors seem to suggest that we're about to assume PA is inconsistent for the corollary, but then the corollary starts by saying "If PA is consistent,..." Is it the "...$\cup\{\neg Con_{PA}\}$" that says PA is inconsistent? So, the task at hand is to show $PA \cup \{\neg Con_{PA}\}$ is consistent?HETZEL: I know the discussion just prior to Corollary 6.6.4 sounds strange—but, I guess that's the point. The way to read the authors' statement "if we like, assume that PA is *inconsistent*" is to see it as assuming that $\neg Con_{PA}$ is true. As such, Corollary 6.6.4 is really saying that granting the consistency of PA allows you to (strangely) conclude that the set of axioms given by PA appended with an axiom that says PA is inconsistent still yields a consistent system (since Con_{PA} is independent of PA by Theorem 6.6.3).

5 Somewhat random questions

I'm just now remembering how in matrix algebra we say a system is inconsistent if the reduced echelon form has a row of zeros ending in a nonzero number. For instance,

$$\left[\begin{array}{cc|c} 3 & 27 & 9 \\ 2 & 18 & 7 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 9 & 3 \\ 2 & 18 & 7 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 9 & 3 \\ 0 & 0 & 1 \end{array}\right]$$

is an inconsistent system of equations because the last row is saying $0x+0y = 0 = 1$.
Does this directly parallel the discussion of consistency and inconsistency in this book? In other words, are we in a way taking $3x + 27y = 9$ and $2x + 18y = 7$ as axioms and seeing whether that set of axioms is consistent or not? HETZEL: Not exactly. However, since $0 = 1$ is the "canonical" logical contradiction in systems such as $\mathcal{L}_{NT}$, the terminology was borrowed. Also, the authors have repeatedly said that if a set of axioms can prove $\perp$ (or any contradiction?) that we would then be able to prove anything, which certainly sounds like something that would result from $0 = 1$. But then again, I suppose it may just be borrowed verbiage, since this could be thought of as a regular old contradiction, but it just happened to cross my mind.

Apart from Chapter 7 and the works referenced throughout this book, do you have any other books you can recommend? HETZEL: My favorite is *A Tour Through Mathematical Logic* by Robert Wolf. You can check it out on amazon.com, although you should definitely be able to find it far cheaper than what they're selling it for.