4.2, 4.5

Andrew Lounsbury

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1 4.2, p.107

1. (a) $\underline{S0 + S0 = SS0}$
   Yes: it's atomic

   (b) $\underline{\neg(0 < 0 \vee 0 < S0)}$
   Yes: Let $\alpha :\equiv 0 < 0$ and $\beta :\equiv 0 < S0$.
   α and β are atomic
   $\alpha \vee \beta$ a Σ-formula (case 2)
   $\neg(\alpha \vee \beta)$ a Σ-formula

   (c) $\underline{(\forall x < \overline{17})x < \overline{17}}$
   Yes: case 4 - $(\forall x < t)\alpha$

   (d) $\underline{S0 \cdot S0 = S0 \wedge (\exists y < x)(\exists z < y)y + z = x}$
   Yes: Let $\alpha :\equiv S0 \cdot S0 = S0$ and $\beta :\equiv (\exists y < x)(\exists z < y)y + z = x$.
   α is atomic
   β is a Σ-formula by case 4
   $\alpha \wedge \beta$ is a Σ-formula by case 3

   (e) $\underline{(\forall y)(y < 0 \to 0 = 0)}$
   No: unbounded universal quantifier

   (f) $\underline{(\exists x)(x < x)}$
   Yes: case 4

2. A formula is Cool if and only if it is a Σ-formula.

   Proof. ∎

3. $\underline{\alpha :\equiv x < y \vee (\ \forall z < w)x + \overline{17} = \overline{42}}$

   (a) Yes: Let $\beta :\equiv x < y$ and $\delta :\equiv (\ \forall z < w)x + \overline{17} = \overline{42}$.
   β is atomic
   $x + \overline{17} = \overline{42}$ is atomic, so δ is a Π-formula
   $\beta \vee \delta$ is a Π-formula by case 3

   (b) $\neg\alpha :\equiv y \leq x \wedge \neg(\ \forall z < w)x + \overline{17} = \overline{42}$
   Yes: Let $\beta :\equiv y \leq x$ and $\delta :\equiv \neg(\ \forall z < w)x + \overline{17} = \overline{42}$.
   β is a Π-formula
   $x+\overline{17} = \overline{42}$ is a Π-formula, so $(\ \forall z < w)x+\overline{17} = \overline{42}$ is a Π-formula
   δ is a Π-formula by case 2
   $\beta \wedge \delta$ is a Π-formula by case 3

   (c) $\neg\alpha :\equiv y \leq x \wedge (\exists z < w)x + \overline{17} \neq \overline{42}$

   (d) If α is any Σ-formula, then $\neg\alpha$ is logically equivalent to a Π-formula.

   Proof. We induct on the complexity of α.
   If α is atomic, then $\neg\alpha$ is clearly a Π-formula.
   If $\alpha :\equiv \neg\beta$, where β is an atomic formula, then $\neg\alpha :\equiv \beta$ is clearly a Π-formula.
   Suppose $\alpha :\equiv \beta \wedge \delta$, where β and δ are Σ-formulas. ■

2 4.5, p.112

1. (a) $\langle 3, 0, 4, 2, 1\rangle = 2^4 \cdot 3^1 \cdot 5^5 \cdot 7^3 \cdot 11^2 = 6,225,450,000$

   (b) $(16910355000)_3 = (2^3 \cdot 3^1 \cdot 5^4 \cdot 7 \cdot 11^3)_3 = (\langle 2, 0, 3, 0, 4\rangle)_3 = 3$

   (c) $|16910355000| = |\,\langle 2, 0, 3, 0, 4\rangle\,| = 5$

   (d) $(16910355000)_{42} = 0$

   (e) $\langle 2, 7, 1, 8\rangle \frown \langle 2, 8, 1\rangle = \langle 2, 7, 1, 8, 2, 8, 1\rangle = 2^3{\cdot}3^8{\cdot}5^2{\cdot}7^9{\cdot}11^3{\cdot}13^9{\cdot}17^2$

   (f) $17 \frown 42 = 0$
   2 is the only prime code number.

2. Consider 6: $\langle 6\rangle = 2^7 = 128$, so $|\,\langle 6\rangle\,| = |128| = |2^7| = 1$, but $|6| = |2 \cdot 3| = 2$.