5.2-5.5

Andrew Lounsbury

March 5, 2020

1 5.2, p.119

1. $N \nvdash (\forall x)(\forall y)x + y = y + x$ and $N \nvdash \neg[(\forall x)(\forall y)x + y = y + x]$.

   Proof. We must find an $\mathcal{L}_{NT}$-structure such that $\mathfrak{A} \models N$ and $\mathfrak{A} \not\models \neg[(\forall x)(\forall y)x + y = y + x]$.
   Section 2.8.1 8: There exists such a $\mathfrak{A}$.
   Proof of 8: Let $\mathfrak{A}$ be a structure with $0^{\mathfrak{A}} = 0$, the underlying set $A = \mathbb{N} \cup \{\alpha, \beta\}$, where α and β are nonstandard, and the following functions and relation

$$S^{\mathfrak{A}}(x) = \begin{cases} x+1 & x \in \mathbb{N} \\ x & x \in \{\alpha, \beta\} \end{cases}$$

$$x +^{\mathfrak{A}} y = \begin{cases} x+y & x, y \in \mathbb{N} \\ x & x \in \{\alpha, \beta\} \text{ and } y \in \mathbb{N} \\ y & o/w \end{cases}$$

$$x \cdot^{\mathfrak{A}} y = \begin{cases} x \cdot y & x, y \in \mathbb{N} \\ x & x \in \{\alpha, \beta\} \text{ and } y \in \mathbb{N} \\ y & o/w \end{cases}$$

   (this one is a bit tricky)

$$xE^{\mathfrak{A}}y = \begin{cases} x^y & x, y \in \mathbb{N} \\ 0 & x \in \{\alpha, \beta\} \text{ and } y \in \mathbb{N} \\ 1 & o/w \end{cases}$$

$$x <^{\mathfrak{A}} y \quad \begin{array}{l} \text{when } x, y \in \mathbb{N} \text{ and } x < y \\ \text{or when } x \in \{a, b\} \text{ and } y \in \mathbb{N} \end{array}$$

Specifically, $(\forall x)\neg x <^{\mathfrak{A}} x$ is false.
To save some time typing and because the first four axioms are covered in the text, we will just show $\mathfrak{A} \vDash \{N_5, \dots, N_{11}\}$. (I'm not entirely sure why they're able to ignore them in the text. It is necessary to consider them in order find an $\mathcal{L}_{NT}$-structure that doesn't model commutativity of addition, isn't it?) HETZEL: Yes it is. I assume they omitted it because the process can be left to the reader.
$\underline{N_5}$: Clear.
$\underline{N_6}$: Clear when $x, y \in \mathbb{N}$.
Let $x \in \{\alpha, \beta\}$ and $y \in \mathbb{N}$. Then, $x \cdot^{\mathfrak{A}} S^{\mathfrak{A}} y = x \cdot^{\mathfrak{A}} (y+1) = x$ and $(x \cdot^{\mathfrak{A}} y) +^{\mathfrak{A}} x = x +^{\mathfrak{A}} x = x$.
$\underline{N_7}$: Clear when $x, y \in \mathbb{N}$.
Let $x \in \{\alpha, \beta\}$ and $y \in \mathbb{N}$. Then, $xE^{\mathfrak{A}}0 = 1$ and $S^{\mathfrak{A}}0 = 1$.
$\underline{N_8}$: Clear when $x, y \in \mathbb{N}$.
Let $x \in \{\alpha, \beta\}$ and $y \in \mathbb{N}$. Then, $xE^{\mathfrak{A}}(S^{\mathfrak{A}}y) = xE^{\mathfrak{A}}(y+1) = 0$ and $(xE^{\mathfrak{A}}y) \cdot^{\mathfrak{A}} x = 0 \cdot^{\mathfrak{A}} x = 0$.
$\underline{N_9}$: Clear.
$\underline{N_{10}}$: In English this says x is less than $y+1$ when x is less than or equal to y, so this is clear when $x, y \in \mathbb{N}$. Since our nonstandard α and β are less than any $y \in \mathbb{N}$, this is also clear when $x \in \{\alpha, \beta\}$ and $y \in \mathbb{N}$ and otherwise.
$\underline{N_{11}}$: Following our definition of $<^{\mathfrak{A}}$, this is fairly clear as well.
Thus $\mathfrak{A}$ is an $\mathcal{L}_{NT}$ structure such that $\mathfrak{A} \nvDash (\forall x)(\forall y)x + y = y + x$ since we defined $<^{\mathfrak{A}}$ such that $(\forall x)x <^{\mathfrak{A}} x$ is not true. $\square$
Now, by the Soundness Theorem, we have $N \nvdash (\forall x)(\forall y)x + y = y + x$. On the other hand $\mathfrak{N}$ is structure such that $\mathfrak{N} \vDash N$ and $\mathfrak{N} \nvDash \neg[(\forall x)(\forall y)x + y = y + x]$, so $N \nvdash \neg(\forall x)(\forall y)x + y = y + x$ again by the Soundness Theorem. ■

2 5.3, p.128

2. Suppose $A \subset \mathbb{N}$ is represented by $\phi(x)$ and $B \subset \mathbb{N}$ is represented by $\psi(x)$. Then,

   $\phi(x) \vee \psi(x)$ represents $A \cup B$

(a) *Proof.* Let $a \in A \cup B$. We must show $N \vdash \phi(\overline{a}) \vee \psi(\overline{a})$.
If $a \in A$, then $N \vdash \phi(\overline{a})$ since $\phi(x)$ represents A.

$\phi(x), \therefore \phi(x) \vee \psi(x)$
$N \vdash \phi(\overline{a}) \vee \psi(\overline{a})$ by the (PC) rule of inference.
If $b \in A$, then $N \vdash \psi(\overline{a})$ since $\psi(x)$ represents B.
$\psi(x), \therefore \phi(x) \vee \psi(x)$
$N \vdash \phi(\overline{a}) \vee \psi(\overline{a})$ by the (PC) rule of inference.
On the other hand, suppose $a \notin A \cup B$. We must show $N \vdash \neg(\phi(\overline{a}) \vee \psi(\overline{a}))$.
Since $a \notin A$ and $a \notin B$, we know $N \vdash \neg\phi(\overline{a})$ and $N \vdash \neg\psi(\overline{a})$ since $\phi(x)$ and $\psi(x)$ represent A and B respectively.
$\neg\phi(\overline{a})$ and $\neg\psi(\overline{a}), \therefore \neg(\phi(\overline{a}) \vee \psi(\overline{a}))$
Thus, $N \vdash \neg(\phi(\overline{a}) \vee \psi(\overline{a}))$ by the (PC) rule of inference.
Thus, $\phi(x) \vee \psi(x)$ represents $A \cup B$. ■

(b) $\phi(x) \wedge \psi(x)$ represents $A \cap B$

Proof. Let $a \in A \cap B$ so that $a \in A$ and $a \in B$.
$N \vdash \phi(\overline{a})$ and $N \vdash \psi(\overline{a})$ since $\phi(x)$ and $\psi(x)$ represent A and B respectively.
$\phi(x)$ and $\psi(x), \therefore \phi(x) \wedge \psi(x)$
Therefore, $N \vdash \phi(\overline{a}) \wedge \psi(\overline{a})$ by the (PC) rule of inference. ■

(c) $\neg\phi(x)$ represents A^c, the complement of A.

Proof. Let $a \in A^c$ so that $a \notin A$. Then, $N \vdash \neg\phi(\overline{a})$ since $\phi(x)$ represents A. If, on the other hand $a \notin A^c$ so that $a \in A$. Then, $N \vdash \phi(\overline{a})$, or $N \vdash \neg(\neg\phi(\overline{a}))$ since $\phi(x)$ represents A. By Definition 5.3.1, $\neg\phi(x)$ represents A^c. ■

3. Every finite subset of the natural numbers is representable, and every subset of $\mathbb{N}$ whose complement is finite is also representable.

Proof. Lemma 2.8.4: If $a = b$, then $N \vdash \overline{a} = \overline{b}$. If $a \neq b$, then $N \vdash \overline{a} \neq \overline{b}$.
For any $n \in \mathbb{N}$, $\phi(x) :\equiv x = \overline{n}$ represents $\{n\}$.
A finite set is a union of these sets, and by Exercise 2 part (a), any finite set is representable.
By part (c) of Exercise 2, every subset of $\mathbb{N}$ whose complement is finite is also representable. ■

7. *Divides*$(x, y) :\equiv (\exists z)(x \cdot z = y)$

   I'm not entirely sure what sets like Divides contain. Something like Even is easy enough to understand, but not so much with things like Divides and Primepair. Since every prime has an adjacent prime, I would think that all primes would lie in Primepair, but then Primepair=Prime. As for Divides, I can only imagine it contains numbers that...divide another number(?), which would would be all of them, which doesn't make much sense. They don't contain pairs of numbers, do they? HETZEL: Indeed they do. In particular, $(1, 2) \in$ Divides, but $(2, 1) \notin$ Divides.

3 5.5, p.136

2. (a) The sequence of symbol numbers for $= +000$ is $(7, 11, 9, 9, 9)$, and the code is $2^8 3^{12} 5^{10} 7^{10} 11^{10}$.

   (b) The sequence of symbol numbers for $= Ev_1 Sv_2 \cdot Ev_1 v_2 v_1$ is $(7, 17, 2, 11, 4, 15, 17, 2, 4, 2)$, and the code is $2^8 3^{18} 5^3 7^{12} 11^5 13^{16} 17^{18} 19^3 23^5 29^3$.

   (c) The sequence of symbol numbers for $(= 00 \vee (\neg < 00))$ is $(21, 7, 9, 9, 3, 21, 1, 19, 9, 9, 23, 23)$, and the code is $2^{22} 3^8 5^{10} 7^{10} 11^4 13^{22} 17^2 19^{20} 23^{10} 29^{10} 31^{24} 37^{24}$.

   (d) We have $(\exists v_2)(< v_2 0) \equiv (\neg \forall \neg v_2)(< v_2 0)$. The sequence of symbol numbers is $(21, 1, 5, 1, 4, 23, 21, 19, 4, 9, 23)$, and the code is $2^{22} 3^2 5^6 7^2 11^5 13^{24} 17^{22} 19^{20} 23^5 29^{10} 31^{24}$.