1.9-2.4

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1 1.9, p.38

1. For any formulas α and β, $\{\alpha, \alpha \to \beta\} \vDash \beta$.

   Proof. Let $\mathfrak{A}$ be an $\mathcal{L}$-structure. Suppose $\mathfrak{A} \vDash \{\alpha, \alpha \to \beta\}$. So, $\mathfrak{A} \vDash \alpha[s]$ and $\mathfrak{A} \vDash (\alpha \to \beta)[s]$ for every assignment function s into $\mathfrak{A}$. Then, $\mathfrak{A} \nvDash \alpha[s]$ for every s or $\mathfrak{A} \vDash \beta[s]$ for every s. Since $\mathfrak{A} \vDash \alpha[s]$ for every s, it must be the case that $\mathfrak{A} \vDash \beta[s]$ for every s. Therefore, $\{\alpha, \alpha \to \beta\} \vDash \beta$. ■

   Modus ponens

4. (a) If $\vDash (\phi \to \psi)$, then $\phi \vDash \psi$.

   Proof. Suppose $\vDash (\phi \to \psi)$ so that $\mathfrak{A} \vDash (\phi \to \psi)[s]$ for every $\mathfrak{A}$ and every s. Then, $\mathfrak{A} \vDash \phi[s] \implies \mathfrak{A} \vDash \psi[s]$ for every $\mathfrak{A}$ and every s. Hence, for every $\mathfrak{A}, s_1$, and s_2, we have $\mathfrak{A} \vDash \phi[s_1] \implies \mathfrak{A} \vDash \psi[s_2]$. Thus, $\phi \vDash \psi$. ■

   (b) If ϕ is $x < y$ and ψ is $z < w$, then $\phi \vDash \psi$, but $\nvDash (\phi \to \psi)$.

   Proof. To show $\phi \vDash \psi$, suppose $\mathfrak{A} \vDash x < y$ so that $\mathfrak{A} \vDash x < y[s]$ for every s. So, $<^{\mathfrak{A}} = A \times A$, where A is the universe of $\mathfrak{A}$. In other words, $a <^{\mathfrak{A}} b$ holds for any a and b in A, which means $\mathfrak{A} \vDash z < w[s]$ for all s. Hence, $\mathfrak{A} \vDash z < w$. Thus, $\mathfrak{A} \vDash \phi[s] \implies \mathfrak{A} \vDash \psi[s]$ for every $\mathfrak{A}$ and every s, which means $\mathfrak{A} \vDash (\phi \to \psi)[s]$ for every $\mathfrak{A}$. Therefore, $\vDash (\phi \to \psi)$, which gives $x < y \vDash z < w$.
   Now, let $\mathfrak{N}$ (the structure for natural numbers) be the model and

let s be an assignment function into $\mathfrak{N}$ such that $s(x) = s(w) = 0$ and $s(y) = s(z) = 1$. Then, $\mathfrak{N} \not\models x < y$ and $\mathfrak{N} \not\models z < w$, so $\mathfrak{N} \not\models (x < y \to z < w)$. Therefore, $\not\models (x < y \to z < w)$, or $\not\models (\phi \to \psi)$, as desired. ∎

2 2.2,p.47

1. $\Sigma = \{[(A(x) \wedge A(x)) \to B(x,y)], A(x), [B(x,y) \to A(x)]\}$
   ROI: modus ponens

   (a) $A(x),\ A(x) \wedge A(x),\ (A(x) \wedge A(x)) \to B(x,y),\ B(x,y)$
   Not a deduction: $(A(x) \wedge A(x)) \to A(x)$ is not in Σ, and $A(x) \wedge A(x)$ (probably?) cannot be deduced given $A(x)$ alone.

   (b) $B(x,y) \to A(x),\ A(x),\ B(x,y)$
   Not a deduction: We must have $B(x,y)$ in order to deduce $A(x)$, and we must have $(A(x) \to B(x,y))$ is in Σ in order to deduce $B(x,y)$.

   (c) $(A(x) \wedge A(x)) \to B(x,y),\ B(x,y) \to A(x),\ (A(x) \wedge A(x)) \to A(x)$
   This is a deduction.

4. $\mathcal{L}$ is $\{R^1\}$
   $B = \{R(x_1),\ R(x_1) \to R(x_2),\ R(x_2) \to R(x_3), \ldots, R(x_i) \to R(x_{i+1})\}$
   ROI: modus ponens

   $B \vdash R(x_j$ for each natural number $j \geq 1$.

   Proof. **Base case:** We have $B \vdash R(x_1)$ and $B \vdash (R(x_1) \to R(x_2))$. Thus, $B \vdash R(x_2)$.
   Induction step: Suppose $B \vdash R(x_k)$ for some $k \in \mathbb{N}$. By our definition of B, we have that $B \vdash (R(x_k) \to R(x_{k+1})$. Hence, $B \vdash R(x_{k+1})$. Therefore, $B \vdash R(x_i)$ for every natural number $j \geq 1$. ∎

3 2.4, p. 54

3. see notes

5. (a) $D = (\exists x \neg R(x)), \neg\, \forall x R(x), (\ \forall x P(x)) \vee (\ \forall x R(x)),\ \forall x P(x), (\ \forall x P(x)) \to Q(y), Q(y))$

   Considering the deduction D, we see that $\phi :\equiv Q(y)$ is a propositional consequence of Γ.

   (b) The only formula in Γ that might allow us to determine (deduce?) that ϕ is true is $Q(y) \vee x + y < z$, but the assumption (?) of the only other formula we have at our disposal forces $Q(y)$ to be true. We cannot determine whether or not ϕ is true, so ϕ is not a propositional consequence of Γ.

   (c) $D = ((\neg P(x, y, x)) \wedge (\neg x < y), \neg x < y, (x < y) \vee M(w, p), M(w, p))$

   Assuming (?) the formulas of Γ, we have found through the deduction D that $\phi :\equiv \neg M(w, p)$ is false. Thus, ϕ is a propositional consequence of Γ. (1st & 3rd formulas contradict each other?)