5.8 - #5

Andrew Lounsbury

April 1, 2020

1 5.8, p.147

5. If a is a natural number greater than or equal to 1, then $p_a \leq 2^{a^a}$, where p_a is the a-th prime number.

   Proof. Base case: If $a = 1$, then, $p_a = p_1 = 2 \leq 2 = 2^{1^1} = 2^{a^a}$.
   Induction: Suppose $p_k \leq 2^{k^k}$ for some arbitrary natural number k. Note that $p_{k+1} \leq p_{\min}$, where $p_{\min}$ is the smallest prime factor of $p_1 \cdots p_k - 1$. Also, note that $p_1 \cdots p_k = 2p_2 \cdots p_k$ is even, so $p_1 \cdots p_k - 1$ is odd. Hence, $p_{k+1} < 3 \leq p_{\min}$. We know $p_{k+1} < 3$ because we would otherwise have $3 \leq p_{k+1} < p_{\min}$, contradicting $p_{k+1} \leq p_{\min}$. Then, $p_{k+1} < 3 < 2^{2^2} \leq 2^{(k+1)^{(k+1)}}$, as desired. ■

I still don't think you have it. Remember that p_a is the ath prime, so there's no way you could reasonably assert that $p_{k+1} < 3$. Consider this for your induction step:

Induction: Suppose $p_k \leq 2^{k^k}$ for each $k \leq n$, where n is some arbitrary natural number. Then

$$\begin{aligned}
p_{k+1} &\leq p_1 p_2 \cdots p_k - 1 \\
&\leq p_1 p_2 \cdots p_k \\
&\leq 2^1 2^{2^2} 2^{3^3} \cdots 2^{k^k} \\
&\leq 2^{k^k} 2^{k^k} \cdots 2^{k^k} \\
&= 2^{k \cdot k^k} \\
&= 2^{k^{k+1}} \\
&\leq 2^{(k+1)^{k+1}}
\end{aligned}$$

as deired.