1.6-1.8

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1 1.6, p.26

3. $\mathcal{L}$ is $\{\flat, \sharp^3, \natural^2\}$

   $\{0, \sharp, \natural\}$, where $\sharp : \mathbb{R}^3 \to \mathbb{R}$ is defined as $\sharp(x, y, z) = x + y + z$ and $\natural$ is defined as $<$, is a model for $\mathcal{L}$.

   $\{0, \sharp, \natural\}$, where $\sharp : \{0, 1, 2, 3, 4, 5\}^3 \to \{0, 1, 2, 3, 4, 5\}$ is defined as $\sharp(x, y, z) = x + y + z$ and $\natural$ is defined as $<$, is a model for $\mathcal{L}$.

7. $\mathcal{L}_{NT}$ is $\{0, S, +, \cdot, E, <\}$
   Let $\mathfrak{A} = (\mathbb{N}, 0^{\mathfrak{A}}, S^{\mathfrak{A}}, +^{\mathfrak{A}}, \cdot^{\mathfrak{A}}, E^{\mathfrak{A}}, <^{\mathfrak{A}})$ be an $\mathcal{L}_{NT}$-structure defined as follows

$$\begin{gathered} S^{\mathfrak{A}}(t) :\equiv St \\ +^{\mathfrak{A}}(t, s) :\equiv +ts \\ \cdot^{\mathfrak{A}}(t, s) :\equiv \cdot ts \\ E^{\mathfrak{A}}(t, s) :\equiv Ets \\ <^{\mathfrak{A}}(t, s) :\equiv < ts \end{gathered}$$

   Let $\phi :\equiv S0 + S0 + SS0$ and let s be an assignment function into $\mathfrak{A}$. Then,

$$\begin{aligned} \overline{s}(S0 + S0) \text{ is } & +^{\mathfrak{A}}(S^{\mathfrak{A}}(0^{\mathfrak{A}}), S^{\mathfrak{A}}(0^{\mathfrak{A}})) \\ \text{is } & +^{\mathfrak{A}}(S^{\mathfrak{A}}(0), S^{\mathfrak{A}}(0)) \\ \text{is } & +^{\mathfrak{A}}(S0, S0) \\ \text{is } & +S0S0, \end{aligned}$$

and

$$\begin{aligned}\overline{s}(SS0) &\text{ is } S^{\mathfrak{A}}(S^{\mathfrak{A}}(0^{\mathfrak{A}}))\\ &\text{ is } S^{\mathfrak{A}}(S^{\mathfrak{A}}(0))\\ &\text{ is } S^{\mathfrak{A}}(S0)\\ &\text{ is } SS0.\end{aligned}$$

Since $\overline{s}(S0 + S0)$ is not the same as $\overline{s}(SS0)$, $\mathfrak{A} \not\models \phi[s]$ and ϕ is not true in $\mathfrak{A}$.

2 1.7, p.32

1. The structure $\mathfrak{N}$ makes the sentence $\phi :\equiv 1 + 1 = 2$ true.

Proof. Let s be an assignment function into $\mathfrak{N}$.

$$\begin{aligned}\overline{s}(1+1) &\text{ is } +^{\mathfrak{N}}(1^{\mathfrak{N}}, 1^{\mathfrak{N}})\\ &\text{ is } 2.\end{aligned}$$

and

$$\begin{aligned}\overline{s}(2) &\text{ is } 2^{\mathfrak{N}}\\ &\text{ is } 2,\end{aligned}$$

Since $\overline{s}(1 + 1)$ is the same as $\overline{s}(2)$, $\mathfrak{N} \models \phi[s]$ and ϕ is true in $\mathfrak{N}$. ■

The structure $\mathfrak{A} = (\mathbb{N}, 0, S, +, \cdot, E, <)$, where $+$ is defined as $+(t, s) = t^2 + s + 1$ makes $\phi :\equiv 1 + 1 = 2$ false.

Proof. Let s be an assignment function into $\mathfrak{A}$.

$$\begin{aligned}s(1+1) &\text{ is } +^{\mathfrak{A}}(1^{\mathfrak{A}}, 1^{\mathfrak{A}})\\ &\text{ is } 1^2 + 1 + 1\\ &\text{ is } 3\end{aligned}$$

and

$$\begin{aligned}s(2) &\text{ is } 2^{\mathfrak{A}}\\ &\text{ is } 2\end{aligned}$$

Since $\overline{s}(1 + 1)$ is the same as $\overline{s}(2)$, $\mathfrak{N} \not\models \phi[s]$ and ϕ is false in $\mathfrak{A}$. ■

The structure $\mathfrak{N}$ makes the sentence $\phi :\equiv (\forall x)(x+1=x)$ false.

Proof. Let s be an assignment function into $\mathfrak{N}$. Then,

$$\begin{aligned}\overline{s}[x|n](x+1) \text{ is } & +^{\mathfrak{N}}(x, 1^{\mathfrak{N}}) \\ \text{is } & x+1\end{aligned}$$

for each $n \in \mathbb{N}$, and

$$\overline{s}[x|n](x) = x$$

for each $n \in \mathbb{N}$. Since $\overline{s}[x|n](x+1)$ and $\overline{s}[x|n](x)$ are not the same for every $n \in \mathbb{N}$, $\mathfrak{N} \not\models \phi[s(x|n)]$ and ϕ is not true in $\mathfrak{N}$. ■

The structure $\mathfrak{A} = (\mathbb{N}, 0, S, +, \cdot, E, <)$, where $+$ is defined as $+(t,s) = ts$, makes $\phi :\equiv (\forall x)(x+1=x)$ true.

Proof. Let s be an assignment function into $\mathfrak{N}$. Then,

$$\begin{aligned}\overline{s}[x|n](x+1) \text{ is } & +^{\mathfrak{A}}(x, 1^{\mathfrak{A}}) \\ \text{is } & x,\end{aligned}$$

for each $n \in \mathbb{N}$, and

$$\overline{s}[x|n](x) = x$$

for each $n \in \mathbb{N}$. Since $\overline{s}[x|n](x+1)$ and $\overline{s}[x|n](x)$ are the same for every $n \in \mathbb{N}$, $\mathfrak{A} \models \phi[s(x|n)]$ and ϕ is true in $\mathfrak{A}$. ■

5. Let $\mathfrak{A}$ be a structure for the language of set theory, $\mathcal{L}_{ST}$, which is $\{\in\}$. Let $A = \{u, v, w, \{u\}, \{u,v\}, \{u,v,w\}\}$. Then, the sentence $\phi :\equiv (\forall y \in y)(\exists x \in x)(x = y)$ is false in $\mathfrak{A}$.

$$\begin{aligned}\mathfrak{A} \models \phi[s] \text{ iff } & \text{For every } a \in A, \mathfrak{A} \models \neg(\forall x \in x)\neg(x=y)[s(y|a)] \\ \text{iff } & \text{For every } a \in A, \mathfrak{A} \not\models (\forall x \in x)\neg(x=y)[s(y|a)] \\ \text{iff } & \text{For every } a \in A, \text{ there is a } b \in A, \text{ such that} \\ & \mathfrak{A} \not\models (\forall x \in x)\neg(x=y)[s(y|a)(x|b)]\end{aligned}$$

Proof. Let s be an assignment function into $\mathfrak{A}$.
Write ϕ as

$$\begin{aligned}
\phi &:\equiv (\forall y \in y)(\exists x \in x)(x = y) \\
&:\equiv (\forall y \in y)\neg(\forall x \in x)\neg(x = y) \\
&:\equiv (\forall y \in y)\neg(\forall x \in x)\neg(\alpha) \\
&:\equiv (\forall y \in y)\neg(\forall x \in x)(\beta) \\
&:\equiv (\forall y \in y)\neg(\gamma) \\
&:\equiv (\forall y \in y)(\delta).
\end{aligned}$$

We have that $\overline{s}(x) = x$ and $\overline{s}(y) = y$. Since these are not the same, $\mathfrak{A} \not\models \alpha[s]$, which means $\mathfrak{A} \models \beta[s]$.
Then, we have that $\overline{s}[x|a](x) = x$ for every $a \in A$ and $\overline{s}[x|b](y) = b$ for every $b \in A$. Since these are not the same for every $b \in A$, we know $\mathfrak{A} \not\models \gamma[s(x|a)]$, which tells us that $\mathfrak{A} \models \delta[s]$.
Now, $\overline{s}[y|a](x) = a$ for every $a \in A$ and $\overline{s}[y|b](y) = y$ for every $b \in A$. Since these are not the same, $\mathfrak{A} \not\models \phi[s]$. Thus, ϕ is false in $\mathfrak{A}$. ■

3 1.8, p.36

2. (a) $\phi :\equiv \forall x(x = y \to Sx = Sy)$, t is $S0$
   $\phi^x_t :\equiv \forall x(t = y \to Sx = Sy)$
   (b) $\phi :\equiv \forall y(x = y \to Sx = Sy)$, t is Sy
   $\phi^x_t :\equiv \forall y(x = y \to St = Sy)$
   (c) $\phi :\equiv x = y \to (\forall x)(Sx = Sy)$, t is Sy
   $\phi^x_t :\equiv x = y \to (\forall x)(St = Sy)$

3. If t is variable-free, then t is always substitutable for x in ϕ.

   Proof. We induct on the complexity of ϕ.
   Base case: Suppose ϕ is atomic. By the definition of substitutability, t is substitutable in ϕ.
   is Suppose t is substitutable for x in formulas α and β.
   If ϕ is of the form $\neg(\alpha)$ or $(\alpha \vee \beta)$, then t is substitutable for x in ϕ by definition given in clauses (2) and (3) of Definition 1.8.3, respectively.
   If $\phi :\equiv (\forall y)(\alpha)$, then t is substitutable for x in ϕ by the fact that t does not contain the variable y and t is substitutable for x in α. ■