5.8 - #5

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1 5.8, p.147

5. If a is a natural number greater than or equal to 1, then $p_a \leq 2^{a^a}$, where p_a is the a-th prime number.

   Proof. Base case: If $a = 1$, then, $p_a = p_1 = 2 \leq 2 = 2^{1^1} = 2^{a^a}$.
   Induction: Suppose $p_k \leq 2^{k^k}$ for some arbitrary natural number k. Note that $p_{k+1} \leq p_{\min}$, where $p_{\min}$ is the smallest prime factor of $p_1 \cdots p_k - 1$. Also, note that $p_1 \cdots p_k = 2p_2 \cdots p_k$ is even, so $p_1 \cdots p_k - 1$ is odd. Hence, $p_{k+1} < 3 \leq p_{\min}$. We know $p_{k+1} < 3$ because we would otherwise have $3 \leq p_{k+1} < p_{\min}$, contradicting $p_{k+1} \leq p_{\min}$. Then, $p_{k+1} < 3 < 2^{2^2} \leq 2^{(k+1)^{(k+1)}}$, as desired. ∎