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ON THE SEPARATING IDEALS OF SOME VECTOR-VALUED GROUP ALGEBRAS

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Abstract. For a locally compact Abelian group G, and a commutative Banach algebra B, let $L^1(G,B)$ be the Banach algebra of all Bochner integrable functions. We show that if G is noncompact and B is a semiprime Banach algebras in which every minimal prime ideal is contained in a regular maximal ideal, then $L^1(G,B)$ contains no nontrivial separating ideal. As a consequence we deduce some automatic continuity results for $L^1(G,B)$.

1. INTRODUCTION. For any locally compact Abelian group G, and commutative Banach algebra B, let $L^1(G,B)$ denote the convolution algebra of all integrable functions on G with values in B. As one might expect, there are some interesting similarities between B and $L^1(G,B)$. For instance, $L^1(G,B)$ is semi-simple if and only if B is semi-simple, and the regular maximal ideals of $L^1(G,B)$ are closely related in a natural way with the regular maximal ideals of both $L^1(G,B)$ and B. Also, $L^1(G,B)$ is Tauberian if and only if B is Tauberian. Refer to [8,9] for the proofs of the above results. Also it is easy to note that $L^1(G,B)$ is semiprime when B is semiprime. The question whether the zero ideal is the only separating ideal in a semiprime Banach algebra still seems to be open. However, in this paper we prove that when G is a noncompact locally compact Abelian group, and B is a commutative semiprime Banach algebra (not necessarily unital) in which every minimal prime ideal is contained in a regular maximal ideal, then $L^1(G,B)$ contains no non-trivial separating ideal. As a consequence we deduce some automatic continuity results for the algebra $L^1(G,B)$. Our results extend some of the results in [11] for non unital Banach algebras, and also extend some results in [7] for semiprime Banach algebras. For relevant information on $L^1(G,B)$ and for related results in harmonic analysis on Abelian groups,

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see [5,8,9,12].

2. PRELIMINARIES. Let B be a commutative Banach algebra (not necessarily unital), and let G be a locally compact Abelian group with Haar measure m. Throughout the following, the dual group of G is denoted by Γ and the spectrum of B is denoted by $\Delta(B)$. Let $L^1(G,B)$ denote the Banach algebra of all integrable function from G into B,

$$(f * g)(t) := \int_G f(t-s)g(s)dm \text{ for all } f,g \in L^1(G,B) \text{ and } t \in G,$$

and let $\|f\|_1 := \int_G \|f(t)\|dm(t)$ for all $f \in L^1(G,B)$. Recall that for any $f \in L^1(G,B)$, and γ in the dual group Γ of G, $\hat{f}(\gamma) = \int_G \overline{\gamma(t)}f(t)dm(t)$ is known as the vector-valued Fourier transform of f at γ. Furthermore for any $\gamma \in \Gamma$, let $M_\gamma := \{f \in L^1(G,B) : \hat{f}(\gamma) = \theta\}$ where θ is the zero vector of B. Clearly, M_γ is a closed ideal of $L^1(G,B)$. If B has no non-trivial zero divisors, then M_γ is a closed prime ideal of $L^1(G,B)$. Recall that an ideal I of a commutative Banach algebra is said to be prime if the product $xy \in I$ only if either $x \in I$ or $y \in I$. It is an easy consequence of the Hahn-Banach theorem that $\bigcap_{\gamma\in\Gamma} M_\gamma$ is the zero ideal in $L^1(G,B)$. For any $\gamma \in \Gamma$, $\phi \in \Delta(B)$, let

$$M_{\gamma,\phi} := \{f \in L^1(G,B) | \phi(\hat{f}(\gamma)) = 0\}.$$

The regular maximal ideals of $L^1(G,B)$ are given by $M_{\gamma,\phi}$ for some $\gamma \in \Gamma$, and $\phi \in \Delta(B)$ ([8]).

For each $f \in L^1(G)$, and $x \in B$, we let

$$(f \otimes x)(s) = f(s)x \text{ for all } s \in G.$$

We recall some of the properties of the product $f \otimes x$ in the following proposition.

Proposition 2.1. Let G be a locally compact Abelian group, and let B be a commutative Banach algebra. Let $x, y \in B$; $f, g \in L^1(G)$; and γ a non-trivial continuous character on G. Then,

(i) $f \otimes x \in L^1(G, B)$, and $\|f \otimes x\|_1 = \|f\|_1 \|x\|$

(ii) $(f \pm g) \otimes x = f \otimes x \pm g \otimes x$

(iii) $\widehat{f \otimes x}(\gamma) = \hat{f}(\gamma)x$

(iv) $(f \otimes x) * (g \otimes x) = (f * g) \otimes xy$

(v) If B has the multiplicative identity 1, then $(f*g)\otimes x = (f\otimes x)*(g\otimes 1) = (f\otimes 1)*(g\otimes x)$

(vi) If $f_n \to f$ in $L^1(G)$ and $x_n \to x$ in B, then $f_n \otimes x_n \to f \otimes x$ in $L^1(G, B)$.

3. Main Results

Before we get to the main results, we need the following lemmas.

Lemma 3.1. Let G be a noncompact locally compact Abelian group, B a commutative Banach algebra, and f a non-zero function in $L^1(G, B)$. For a given γ in the dual group Γ of G and a positive number ε, there exist $f_1, f_2, \ldots, f_n$ in $L^1(G)$ with compactly supported Fourier transforms and $x_1, x_2, \ldots, x_n$ in B such that $\|f - \sum_{i=1}^{n} f_i \otimes x_i\| < \epsilon + \|\hat{f}(\gamma)\|$, where $\hat{f}_i(\gamma) = 0$ for $1 \le i \le n$.

Proof. Since finite linear combinations of the elements of the form $h \otimes x$ where $h \in L^1(G)$, and $x \in B$ are dense in $L^1(G)$, and the functions in $L^1(G)$ with compactly supported Fourier transforms are dense in $L^1(G)$, there exist $h_1, h_2, \ldots, h_n$ in $L^1(G)$ with compactly supported Fourier transforms and $x_1, x_2, \ldots, x_n$ in B such that $\|f - \sum_{i=1}^{n} h_i \otimes x_i\| < \frac{\varepsilon}{2}$. For $1 \le i \le n$, let $Supp\ \hat{h}_i = \{\alpha \in \Gamma : \hat{h}_i(\alpha) \ne 0\}$. For each $1 \le i \le n$, we define

$$g(t) = \frac{\chi_{(\cup_{j=1}^{n} Supp\ \hat{h}_j)}}{m(\cup_{j=1}^{n} Supp\ \hat{h}_i)} \gamma(t),$$

where $\chi_{(\cup_{j=1}^{n} Supp\ \hat{h}_j)}$ is the characteristic function of $(\cup_{j=1}^{n} Supp\ \hat{h}_i)$, and $f_i = h_i - \hat{h}_i(\gamma)g$. Clearly g and the f_i's belong to $L^1(G)$. It is easy to see that $\hat{g}(\gamma) = 1$, $\hat{f}_i(\gamma) = 0$ for each

i, and $\|g\|_1 = 1$. We have

$$\|f - \sum_{i=1} (f_i \otimes x_i) - g \otimes \hat{f}(\gamma)\|$$

$$= \|f - \sum_{i=1} (h_i \otimes x_i) + \sum_{i=1} (h_i \otimes x_i) - \sum_{i=1} (f_i \otimes x_i) - g \otimes \hat{f}(\gamma)\|$$

$$\leq \|f - \sum_{i=1} (h_i \otimes x_i)\| + \|\sum_{i=1} (h_i - f_i) \otimes x_i - g \otimes \hat{f}(\gamma)\| \quad \ldots (A)$$

Furthermore,

$$\|\sum_{i=1} (h_i - f_i) \otimes x_i - g \otimes \hat{f}(\gamma)| = \|\sum_{i=1} \hat{h}_i(\gamma) g \otimes s_i - g \otimes \hat{f}(\gamma)\|$$

$$= \int_G \|\sum_{i=1} \hat{h}_i(\gamma) g(t) x_i - g(t)\hat{f}(\gamma)\| dm(t)$$

$$= \frac{1}{m(\cup_{j=1}^n Supp\ \hat{h}_j)} \int_G \|\sum_{i=1} \hat{h}_i(\gamma) \chi_{(\cup_{j=1}^n Supp\ \hat{h}_j)} \gamma(t) x_i - \chi_{(\cup_{j=1}^n Supp\ \hat{h}_j)} \gamma(t) \hat{f}(\gamma)\| dm(t)$$

$$= \frac{1}{m(\cup_{j=1}^n Supp\ \hat{h}_j)} \int_{(\cup_{j=1}^n Supp\ h_j)} \|\sum_{i=1} \hat{h}_i(\gamma) x_i - \hat{f}(\gamma)\| dm(t)$$

$$= \|\sum_{i=1} \hat{h}_i(\gamma) x_i - \hat{f}(\gamma)\| \leq \|f - \sum_{i=1} h_i \otimes x_i\| < \frac{\varepsilon}{2} \quad \ldots (B)$$

From (A) and (B) it follows that $\|F - \sum_{i=1} f_i \otimes x_i - g \otimes \hat{f}(\gamma)\| < \epsilon$. Hence $\|f - \sum_{i=1} f_i \otimes x_i\| < \epsilon + \|\hat{f}(\gamma)\|$. This completes the proof of the Lemma. ¥

Lemma 3.2. Let G be a noncompact locally compact Abelian group, B a commutative Banach algebra, and f a non-zero function in $L^1(G, B)$. For a given γ in the dual group Γ of G and a given positive number $\epsilon > 0$, there exist $g_1, g_2, \ldots, g_n$ in $L^1(G)$, a neighborhood V of γ, and $x_1, x_2, \ldots, x_n$ in B such that

$$\|f - \sum_{i=1} g_i \otimes x_i\| < \epsilon + \|\hat{f}(\gamma)\|$$

where $\hat{g}_i = 0$ on V for $1 \leq i \leq n$.

Proof. By Lemma 3.1, there exist $f_1, f_2, \dots, f_n$ in $L^1(G)$ with compactly supported Fourier transforms, and $x_1, x_2, \dots, x_n$ in B such that

$$\|f - \sum_{i=1} f_i \otimes x_i\| < \frac{\epsilon}{2} + \|\hat{f}(\gamma)\|$$

where $\hat{f}_i(\gamma) = 0$. Since $L^1(G)$ satisfies the Ditkin's condition ([12]), there exist $g_1, g_2, \dots, g_n$ in $L^1(G)$, and a neighborhood V of γ such that $\hat{g}_i = 0$ on V, and

$$\|f_i - g_i\|_1 < \frac{\epsilon}{2(1 + \sum_{i=1} \|x_i\|)}$$

for $1 \le i \le n$. Now

$$\begin{aligned}\|f - \sum_{i=1} g_i \otimes x_i\|_1 &\le \|f - \sum_{i=1} f_i \otimes x_i\|_1 + \|\sum_{i=1} (f_i - g_i) \otimes x_i\|_1 \\ &\le \frac{\epsilon}{2} + \|\hat{f}(\gamma)\| + \sum_{i=1} \|f_i - g_i\|_1 \|x_i\| \\ &< \frac{\epsilon}{2} + \|\hat{f}(\gamma)\| + \frac{\epsilon}{2(1 + \sum_{i=1} \|x_i\|)} (\sum_{i=1} \|x_i\|) \\ &= \epsilon + \|\hat{f}(\gamma)\|. \qquad \blacksquare\end{aligned}$$

Corollary 3.3. Let $f \in L^1(G, B)$, and $\gamma \in \Gamma$ such that $\hat{f}(\gamma) = \theta$. Given $\epsilon > 0$, there exist $g_1, g_2, \dots g_n$ in $L^1(G)$ with a vanishing Fourier transform in a neighborhood V of γ, and $x_1, x_2, \dots, x_n$ in B such that $\|f - \sum_{i=1} g_i \otimes x_i\| < \epsilon$.

Proof. Obviously follows from the Lemma 3.2. $\blacksquare$

Now we are ready for the main results of the section.

Theorem 3.4 Let G be a locally compact Abelian group, γ a continuous character on G, and $\mathcal{P}$ a prime ideal contained in M_γ. Then $\mathcal{P}$ is dense in M_γ.

Proof. Let $\mathcal{P}$ be a prime ideal of $L^1(G, B)$ contained in M_γ. Let f be a function with $\hat{f}$ identically equal to the zero vector in a neighborhood V of γ. We claim that f belongs to $\mathcal{P}$. For, if g belongs to $L^1(G)$ with $\hat{g}(\gamma) \neq 0$, $\hat{g} = 0$ on $\Gamma - V$, and x a non-zero vector in B, then $(g \otimes x) * f = \Theta$ (the zero vector of $L^1(G, B)$). Since $\mathcal{P}$ is a prime ideal of $L^1(G, B)$, either $g \otimes x \in \mathcal{P}$ or $f \in \mathcal{P}$. But $g \hat{\otimes} x(\gamma) = \hat{g}(\gamma)x \neq \theta$. Hence $f \in \mathcal{P}$. Thus all the functions f in $L^1(G, B)$ with vanishing Fourier transforms in a neighborhood of γ belong to $\mathcal{P}$. Hence by Lemma 3.2, it follows that $\mathcal{P}$ is dense in M_γ. This completes the proof of the theorem. ¥

Theorem 3.5. Let G be a noncompact locally compact Abelian group, and B be a commutative Banach algebra. If $\mathcal{P}$ is a closed prime ideal of $L^1(G, B)$ contained in $M_{\gamma,\phi}$ for some $\gamma \in \Gamma$, and $\phi \in \Delta(B)$, then $\mathcal{P}$ contains M_γ. Furthermore $\mathcal{P}$ does not contain M_σ for any $\sigma \neq \gamma$.

Proof. Let $f \in M_\gamma$. By Corollary 3.3, f can be approximated by a function g in $L^1(G, B)$ with vanishing Fourier transform in a neighborhood V of γ. By an argument similar to the one given in Theorem 3.4, we can show $g \in \mathcal{P}$. Since $\mathcal{P}$ is a closed ideal, it follows that $f \in \mathcal{P}$. Thus M_γ is contained in $\mathcal{P}$. Let $\sigma \in \Gamma$ such that $\sigma \neq \gamma$. Suppose V_σ and V_γ are compact neighborhoods of σ and γ respectively such $V_\sigma \cap V_\gamma = \emptyset$. Then there exist functions f_σ and f_γ from G into the complex plane with the support of $\hat{f}_\sigma$ contained in V_σ and the support of $\hat{f}_\gamma$ contained in V_γ such that $\hat{f}_\sigma(\sigma) = 1$ and $\hat{f}_\gamma(\gamma) = 1$. Let $x, y \in B$ such that $\phi(x)\phi(y) \neq 0$. Then $f_\sigma \otimes x,\ f_\gamma \otimes y \in L^1(G, B)$ such that $(f_\sigma \otimes x) * (f_\sigma \otimes y) = \Theta$. Since $\mathcal{P}$ is a prime ideal contained in $M_{\gamma,\sigma}$, we get $f_\sigma \otimes x \in \mathcal{P}$. Obviously $f_\gamma \otimes y \notin \mathcal{P}$. However $f_\gamma \otimes y \in M_\sigma$. Therefore M_σ is not contained in $\mathcal{P}$. ¥

4. Applications.

Recall that a closed ideal S of a commutative Banach algebra A is called a separating ideal ([3]) if it satisfies the following condition: For each sequence $\{a_k\}_{k\geq 1}$ in A there is a positive integer n such that $\overline{a_1 a_2 \cdots a_n S} = \overline{a_1 a_2 \cdots a_k S}$ $(k \geq n)$. For any derivation D on A, let $\Im(D) =: \{a \in A |$ there is a sequence $\{a_n\}$ in A with $a_n \to 0$ and $Da_n \to a\}$. For any epimorphism h form a commutative Banach algebra X onto A, let $\Im(h) =: \{a \in A|$ there is a sequence $\{x_n\}$ in X with $x_n \to 0$ and $h(x_n) \to a\}$. It is easy to show that $\Im(D)$,and $\Im(h)$ are closed ideals of A. By the closed graph theorem D is continuous if and only if $\Im(D)$ is zero. Similarly h is continuous if and only if $\Im(h)$ is zero. It is well known that $\Im(D)$ and $\Im(h)$ are separating ideals of A ([13]). For further information on separating ideals, their relation to the prime ideals of the Banach algebra, and for related results on automatic continuity theory, see [1,2,3,4,6,10].

Now we are ready to state one of the main results of the section.

Theorem 4.1. Let G be a noncompact locally compact Abelian group G, and B a commutative semiprime Banach algebra in which every minimal prime ideal is contained in a regular maximal ideal. Then $L^1(G,B)$ contains no nontrivial separating ideal.

Lemma 4.2. Let G be a noncompact locally compact Abelian group G, and B a commutative semiprime Banach algebra. For any $\gamma \in \Gamma$, $M_\gamma = \bigcap_{\mathcal{P}\in\mathcal{I}_\gamma} \mathcal{P}$ where $\mathcal{I}_\gamma$ is the set of all minimal prime ideals of $L^1(G,B)$ containing M_γ.

Proof. Let $f \in \bigcap_{\mathcal{P}\in\mathcal{I}_\gamma} \mathcal{P}$. Since there is a one-to-one correspondence between the prime ideals of the quotient algebra $L^1(G,B)/M_\gamma$ and the prime ideals of the algebra $L^1(G,B)$ containing M_γ, there exists a positive integer n such that $\underbrace{f * f * \cdots * f}_{n \text{ times}} \in M_\gamma$. This implies $(\hat{f}(\gamma))^n = \theta$. Since B is semiprime, $\hat{f}(\gamma) = \theta$. Hence $f \in M_\gamma$. ¥

Proof of Theorem 4.1. If possible assume that $\Im$ is a nontrivial separating ideal in $L^1(G,B)$.

Claim. $\Im$ is contained in all but finitely many M_γ for $\gamma \in \Gamma$.

Proof of the claim. Let $\mathcal{M}$ be the set of all minimal prime ideals of $L^1(G,B)$ not containing $\Im$. By [3] $\mathcal{M}$ is a finite set. Let

$$\mathcal{M}_\Delta = \{\mathcal{P} \in \mathcal{M} | \mathcal{P} \subseteq M_{\gamma,\phi} \text{ for some } (\gamma,\phi) \in \Gamma \times \Delta(B)\}$$

and $\mathcal{M}_{\Delta^0} = \mathcal{M} - \mathcal{M}_\Delta$. By Theorem 3.5, each member of $\mathcal{M}_\Delta$ contains a unique M_γ for some $\gamma \in \Gamma$. Let $\Gamma_{\mathcal{M}_\Delta} = \{\gamma \in \Gamma | M_\gamma \subseteq \mathcal{P}$ for some $\mathcal{P} \in \mathcal{M}_\Delta\}$. Obviously $\Gamma_{\mathcal{M}_\Delta}$ is a finite set. Since $\Im$ is contained in all but finitely many closed prime ideals of $L^1(G,B)$ ([3]), and since any prime ideal contains a minimal prime ideal, it follows that $\Gamma_{\mathcal{M}_\Delta}$ is not empty. Let $\gamma \in \Gamma - \Gamma_{\mathcal{M}_\Delta}$. By Lemma 4.2, $M_\gamma = \bigcap_{\mathcal{P}\in\mathcal{I}_\gamma} \mathcal{P}$ where $\mathcal{I}_\gamma$ is the set consisting of all minimal prime ideals of $L^1(G,B)$ containing M_γ. Write $\mathcal{I}_\gamma = \mathcal{I}_\Delta \cup \mathcal{I}_{\Delta^0} \cup \mathcal{I}_{\Delta^{00}}$ where

$$\mathcal{I}_\Delta = \{\mathcal{P} \in \mathcal{I}_\gamma | \mathcal{P} \subseteq M_{\gamma,\phi} \text{ for some } \phi \in \Delta(B)\},$$

$$\mathcal{I}_{\Delta^0} = \{\mathcal{P} \in \mathcal{I}_\gamma | \mathcal{P} \text{ contains } \Im, \text{ and } \mathcal{P} \not\subseteq M_{\gamma,\phi} \text{ for each } \phi \in \Delta(B)\}$$

and

$$\mathcal{I}_{\Delta^{00}} = \{\mathcal{P} \in \mathcal{I}_\gamma | \mathcal{P} \text{ does not contain } \Im \text{ and } \mathcal{P} \not\subseteq M_{\gamma,\phi} \text{ for each } \phi \in \Delta(B)\}.$$

Notice that $\mathcal{I}_{\Delta^{00}}$ is almost a finite set, and each $\mathcal{P}$ in $\mathcal{I}_\Delta$ contains $\Im$. Obviously

$$\mathcal{M} = (\bigcap_{\mathcal{P}\in\mathcal{I}_\Delta\cup\mathcal{I}_{\Delta^0}} \mathcal{P}) \cap (\bigcap_{\mathcal{P}\in\mathcal{I}_{\Delta^{00}}} \mathcal{P}).$$

In the above, if $\mathcal{I}_{\Delta^{00}}$ is empty then $\bigcap_{\mathcal{P}\in\mathcal{I}_{\Delta^{00}}} \mathcal{P}$ is taken to be $L^1(G,B)$. Since $\mathcal{I}_{\Delta^{00}}$ is utmost a finite set, and $M_{\gamma,\phi}$ is a prime ideal for each $\phi \in \Delta(B)$, $\bigcap_{\mathcal{P}\in\mathcal{I}_{\Delta^{00}}} \mathcal{P} \not\subseteq M_{\gamma,\phi}$. Let $f \in \bigcap_{\mathcal{P}\in\mathcal{I}_\Delta\cup\mathcal{I}_{\Delta^0}} \mathcal{P}$. Choose $g \in (\bigcap_{\mathcal{P}\in\mathcal{I}_{\Delta^{00}}} \mathcal{P} \backslash M_{\gamma,\phi})$. Then $fg \in M_\gamma$. Since $\phi(\hat{g}(\gamma)) \neq 0$ for each $\phi \in \Delta(B)$, by the assumption on B, $\hat{f}(\gamma)$ belongs to every minimal prime ideal of B. Since B is semiprime, $\hat{f}(\gamma) = \theta$. Thus $M_\gamma = \bigcap_{\mathcal{P}\in\mathcal{I}_\Delta\cup\mathcal{I}_{\Delta^0}} \mathcal{P}$. This implies $\Im \subset M_\gamma$. This completes the proof of the claim.

For the remainder of the proof, the argument is similar to Theorem 3.3 of [7].

Let $\Gamma_{\mathcal{M}_\Delta} = \{\gamma_1, \gamma_2, \cdots, \gamma_n\}$. Let $h \in (G \cap (\cap_{i=2}^n M_{\gamma_i})) \backslash M_{\gamma_1}$. Since there exists a minimal prime ideal $\mathcal{P} \in \mathcal{M}$ contains M_{γ_1} but not any of the M_{γ_i}'s for $2 \leq i \leq n$, such a function h exists. Since $\hat{h}(\gamma_1) \neq \theta$, there exists a continuous linear functional λ on B such that $\lambda(\hat{f}(\gamma_1)) \neq 0$. Consider the basic open set

$$N = \{\gamma \in \Gamma : |\lambda(\hat{h}(\gamma)) - \lambda(\hat{h}(\gamma_1))| < |\lambda(\hat{h}(\gamma_1))|\}$$

of Γ containing γ_1. Since G is a noncompact Abelian group, γ_1 is not an isolated point in Γ. By the choice of h, the characters $\gamma_2, \gamma_3, \cdots, \gamma_n$ do not belong to N. Hence there exists a character $\gamma_0 \in \Gamma \backslash \{\gamma_1, \gamma_2, \cdots, \gamma_n\}$ such that $\gamma_0 \in N$. Since $\Im$ is contained in M_{γ_0}, $\hat{h}(\gamma_0) = \theta$. Hence $|\lambda(\hat{h}(\gamma_1))| = |\lambda(\hat{h}(\gamma_1)) - \lambda(\hat{h}(\gamma_0))| < |\lambda(\hat{h}(\gamma_1))|$. This is a contradiction. Therefore $L^1(G, B)$ does not contain a non-trivial separating ideal. ¥

The following result extends Theorem 3.3 of [7] (which in turn extends Theorem 5 of [11]) to some semiprime Banach algebras which do not posses the multiplicative identity.

Theorem 4.3. Let G be a noncompact locally compact Abelian group, and B be a commutative semiprime Banach algebra in which every minimal prime ideal is contained in a regular maximal ideal. Then every derivation on $L^1(G, B)$ is continuous. Also every epimorphism form a commutative Banach algebra onto $L^1(G, B)$ is continuous.

Proof. Obviously follows from Theorem 4.1 and the closed graph theorem. ¥

Remark. If B has the multiplicative identity then every proper prime ideal is contained in a maximal ideal of B. Even if B does not have the multiplicative identity, in most of the algebras every minimal prime ideal is contained in a regular maximal ideal. Therefore the assumption in the above theorem that every minimal prime ideal contained in a regular maximal ideal of the algebra is not too restrictive.

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