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ON THE CARDINALITIES OF ROW SPACES OF BOOLEAN MATRICES

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On the Cardinalities of Row Spaces of Boolean Matrices

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A Boolean matrix has entries that are 0 or 1. For any $n \in N$, the set of all $n \times n$ Boolean matrices with the operation of matrix multiplication (except that $1+1=1$) is a semigroup with the usual identity. This semigroup is denoted B_n. Green's J-relation (on B_n) is defined as follows: given $A, B \in B_n$, A is said to be *equivalent* to B, if the principal ideal generated by A equals the principal ideal generated by B. The row space of a Boolean matrix is the set of all finite linear combinations of rows of the matrix (where the scalars forming the linear combination are either 0 or 1). We call the cardinality of a row space its row span. Every matrix in a given J-class has the same row span (Markowsky in [1]).

For a given n, row spans lie between 1 (the row span of the zero matrix) and 2^n (the row span of the identity matrix). If one principal ideal of Boolean matrices is properly contained in another, then the row span of the matrices that generate the first ideal is less than the row span of matrices that generate the second ideal (a consequence of Zaretskii's Theorem [5]). Thus, the number of different row spans serves as an upper bound on the length of the longest chain of principal ideals under the partial order of set containment. So a natural, and unanswered, question is "What numbers between 1 and 2^n are the row span of some matrix in B_n?"

Konieczny [2] showed that any row span larger than 2^{n-1} must be of the form $2^{n-1}+2^k$ where $0 \le k \le n-1$. He also showed that for each k there are matrices with that row span. Thus, for any n, there are exactly n numbers larger than 2^{n-1} which are the row span of some $n \times n$ Boolean matrix. For numbers smaller than 2^{n-1}, Wen Li and Mou-cheng Zhang [3] showed for $n \ge 7, 2^{n-1}-1$ is not the row span of any matrix. Recently, Hong Shaogang [4] showed that for $n \ge 8, 2^{n-1}-k, k = 1, 2, 3$ are not the row spans of any matrix.

The purpose of this note is to give evidence that the latter result may be expanded significantly. The authors ran a program to count the J-classes in B_8 and find the row span of a representative matrix in each class. The program was tested on B_n for $3 \le n \le 7$ as a way of verifying the accuracy of its work on B_8. The program found $4,256,203,214$ classes in B_8 and found that the following numbers are not the row span of any matrix in B_8: 109, 111, 117, 119, 121, 122, 123, 125, 126 and 127 (the latter three being what was predicted in [4]).

References

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[4] Shaofong, Hong, Gaps in the Cardinalities of Row Spaces of Boolean Matrices, Semigroup Forum **56** (1998), 158–166.

[5] Zaretskii, K.A., The Semigroup of Binary Relations, Mat. Sb. **61** (1963), 291–305 (Russian).

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