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Hecke Algebra Representations in Ideals Generated by $\mathtt{q}$-Young Clifford Idempotents*

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Abstract

It is a well known fact from the group theory that irreducible tensor representations of classical groups are suitably characterized by irreducible representations of the symmetric groups. However, due to their different nature, vector and spinor representations are only connected and not united in such description.

Clifford algebras are an ideal tool with which to describe symmetries of multiparticle systems since they contain spinor and vector representations within the same formalism, and, moreover, allow for a complete study of all classical Lie groups. In this work, together with an accompanying work also presented at this conference, an analysis of $\mathtt{q}$-symmetry – for generic $\mathtt{q}$'s – based on the ordinary symmetric groups is given for the first time. We construct $\mathtt{q}$-Young operators as Clifford idempotents and the Hecke algebra representations in ideals generated by these operators. Various relations as orthogonality of representations and completeness are given explicitly, and the symmetry types of representations is discussed. Appropriate $\mathtt{q}$-Young diagrams and tableaux are given. The ordinary case of the symmetric group is obtained in the limit $\mathtt{q} \to 1$. All in all, a toolkit for Clifford algebraic treatment of multi-particle systems is provided. The distinguishing feature of this paper is that the Young operators of conjugated Young diagrams are related by Clifford reversion, connecting Clifford algebra and Hecke algebra features. This contrasts the purely Hecke algebraic approach of King and Wybourne, who do not embed Hecke algebras into Clifford algebras.

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1 Introduction

1.1 Motivation

We investigate a possibility of implementing the symmetric group S_n and its group algebra deformation, the Hecke algebra $H_{\mathsf{F}}(n, q)$, as a subalgebra of the Clifford algebra of multivectors. The latter algebra is defined as the Clifford algebra of a bilinear form with a suitably chosen anti-symmetric part. The presence of the antisymmetric part changes the structure of the corresponding Clifford algebra and allows one to introduce the needed deformation.

Our main interest in Clifford algebras arose from their ubiquitous appearance in mathematical physics, as it has been demonstrated many times by D. Hestenes [22, 23]. Up to now, however, the main efforts have been devoted to the development of the real Dirac theory and other physical models such as, for example, the Weinberg–Salam theory [9] of electro-weak interactions. Despite this enormous range of applicability, there exist problems in mathematical physics not yet formulated or discussed in the Clifford algebra framework.

One major unsolved problem is the proper formulation of multi-particle theories. Quantum field theory is a theory of infinitely many particles which causes on one hand great problems with renormalization, but on the other it provides one of the most precise formalisms developed so far in physics. There have been only a few attempts to tackle the multiparticle problem [10], while Hestenes uses matrices in this case [23]. On the other hand, we succeeded in showing that quantum field theory, when treated in terms of generating functionals, can be reformulated by Clifford algebras [12, 13, 17]. However, to treat such complicated theories correctly, one is *forced* to introduce non-symmetric bilinear forms in Clifford algebras and there are at least two reasons why this needs to be done.

One reason is a problem of normal-ordering, which has to be performed in multiparticle quantum systems. The transition from time-ordered to normal-ordered generating functionals usually yields singularities, which can be seen as a calculational error if Clifford algebras are used [18]. The second reason is the connection of the vacuum structure of physical systems, which is intimately related to the antisymmetric part of the bilinear form in the Clifford algebra [14]. This is far beyond the abilities of other currently used methods.

The Clifford approach to such problems is very rigid. Treating Clifford numbers as single entities makes it easy to calculate with them but it hides the internal structure of the involved objects. Since essentially all physical observables can be given in terms of two-spinors [32], we wish to have a mechanism which breaks up the Clifford numbers into smaller parts.

Taking an idea from the group theory, it is possible to characterize tensor products of irreducible representations of classical groups by the irreducible representations of the symmetric group, since both group actions commute. Such a situation, where one has a nontrivial action of S_n, is then considered to be an n-particle system. The symmetric group plays hence a dominant role not only in mathematics, e.g., in combinatorics, but also in physics. As an example note that it was a group theoretical necessity to form mesons and hadrons from quarks after they had been postulated.

During the last two decades one has become aware that the deformed symmetric group algebra, i.e., the Hecke algebra, lays at the heart of the so-called quantized structures, see e.g. [6, 28]. In particular, quantum groups have provided a powerful tool for solving lattice problems in statistical physics. Jones polynomial and the knot theory are related to Hecke algebras, see the extensive discussion in [11]. It is thus a quite natural idea to bring the symmetric group and its deformation, the Hecke algebra, into the Clifford formalism. Furthermore, the symmetric group is the Coxeter group of the Dynkin diagram of the A_n complex Lie algebras [24] which explains the name given to the particular deformation discussed below.

To our knowledge, Clifford algebras have been used only marginally in the study of the symmetric groups. Only a spinor double cover is known, see [25], however, D. Finkelstein discussed this quite extensively in his lecture.

In this paper we show explicit calculations for $H_{\mathsf{F}}(2,q)$ and $H_{\mathsf{F}}(3,q)$ where F is the base field of the group algebra in question. [1]

1.2 Definitions

Throughout this paper, we use Clifford algebras with an arbitrary not necessarily symmetric bilinear form B. Such algebras can be constructed using Chevalley's approach [5]. However, he utilizes only symmetric forms thereafter. This issue was clearly addressed in [1, 16, 29, 30], while the connection to Hecke algebras was first discussed in [31]. A mathematically sound approach to such algebras including their applications can be found in [11, 17]. There is a need imposed by quantum physics to use this type of Clifford algebras as it was shown in [14].

Spinors are usually defined to be elements of a minimal left ideal of a Clifford algebra. Such ideals can be constructed as linear spaces generated by a primitive idempotent. This means, that the spinor space $\mathcal{S} = < C\ell(B,V)\mathsf{f} >_{\mathsf{K}}$, where f is a primitive idempotent and $C\ell(B,V)$ is the full Clifford algebra of the pair (B,V), where B is the general bilinear form and V a linear space over F. It is well known, that the smallest faithful representation of a Clifford algebra is a spinor representation of dimension 2^k, where k is related to a Radon-Hurwitz number. Comparing the dimension of the space of endomorphisms of the spinor space $\mathcal{S}$ with that of the Clifford algebra, one finds several cases of representations over the field $\mathsf{K} \cong \mathsf{R}, \mathsf{C}, \mathsf{H}, \mathsf{R}\oplus\mathsf{R}$ or $\mathsf{H}\oplus\mathsf{H}$. Moreover, since the primitive idempotents of a Clifford algebra decompose the unity $\mathsf{1} = \mathsf{f}_1 + \mathsf{f}_2 + \cdots + \mathsf{f}_k$, one ends up with a k-dimensional right-linear space $\mathcal{S}$ over the appropriate field K.

The aim of this work is to provide a mechanism for breaking up the ordinary spinor representation of $C\ell_{n,n}$ into tensor products of smaller representations using appropriate Young operators constructed as Clifford idempotents. This means, that a suitable Clifford algebra is used as a carrier space for various tensor product representations.

The Young operators for various Young diagrams provide us with a set of idempotents which decompose the unity $\mathsf{1}$ of $C\ell_{n,n}$ as

$$\mathsf{1} = Y^{(\lambda_1)} + \cdots + Y^{(\lambda_n)}, \tag{1}$$

where (λ_i) is a partition of n characterizing the appropriate Young tableau, that is, a Young diagram (frame) with an allowed numbering. We denote a Young tableaux by $Y^{(\lambda_i)}_{i_1,\ldots,i_n}$, where (λ_i) is an ordered partition of n and $i_1,\ldots,i_n$ is an allowed numbering of the boxes in the Young diagram corresponding to (λ_i) as in [21, 27]. Furthermore, these Young operators are mutually annihilating idempotents

$$Y^{(\lambda_i)}Y^{(\lambda_j)} = \delta_{\lambda_i\lambda_j}Y^{(\lambda_j)}. \tag{2}$$

It appears natural to ask if these Young operators can be used to give representations of the symmetric group *within* the Clifford algebraic framework. The representation

[1] We distinguish the field F the algebras are built over and the (double) field K, defined below, used in representations of Clifford algebras. They should not be confused.

spaces which appear as a natural outcome of the embedding of the symmetric group and its representations can then be looked at as multiparticle spinor states. However, these might not be spinors of the full Clifford algebra.

In order to be as general as possible, we give not only the representations of the symmetric group but also of the Hecke algebra $H_{\mathsf{F}}(n,q)$. The Hecke algebra is the generalization of the group algebra of the symmetric group by adding the requirement that transpositions t_i of *adjacent* elements $i, i+1$ are no longer involutions s_i. We set $t_i^2 = (1-q)t_i + q$ which reduces to $s_i^2 = \mathbf{1}$ in the limit $q \to 1$.

Hecke algebras are 'truncated' braids, since a further relation (see (3) below) is added to the braid group relations as in [4]. A detailed treatment of this topic with important links to physics may be found, for example, in [20, 34] and in the references of [11].

The defining relations of the Hecke algebra will be given according to Bourbaki [8]. Let $< \mathbf{1}, t_1, \ldots, t_n >$ be a set of generators which fulfill these relations:

$$\begin{aligned} t_i^2 &= (1-q)t_i + q, && (3)\\ t_i t_j &= t_j t_i, \quad |i-j| \geq 2, && (4)\\ t_i t_{i+1} t_i &= t_{i+1} t_i t_{i+1}, && (5) \end{aligned}$$

then their algebraic span is the Hecke algebra. Since we will compare our results with those of King and Wybourne [26] – hereafter denoted by KW – we give a transformation to their generators g_i, namely $g_i = -t_i$, which results in a new quadratic relation

$$g_i^2 = (q-1)g_i + q \tag{6}$$

while the other two remain unchanged. However, this small change in sign is responsible for great differences especially in the q-polynomials occuring in our formulas and in their formulas. One immediate consequence is that this transformation interchanges symmetrizers and antisymmetrizers. In particular, this replacement connects formula (3.4) in KW with our full symmetrizers while formula (3.3) in KW gives our full antisymmetrizers. Finally, the algebra morphism ρ which maps the Hecke algebra into the even part of an appropriate Clifford algebra can be found in [11].

Let $< \mathbf{1}, \mathsf{e}_1, \ldots, \mathsf{e}_{2n} >$ be a set of generators of the Clifford algebra $C\ell(B,V)$, $V = \mathrm{span}\{\mathsf{e}_i\}$, with a non-symmetric $2n \times 2n$ bilinear form $B = [B(\mathsf{e}_i, \mathsf{e}_j)] = [B_{i,j}]$ defined as

$$B_{i,j} := \begin{cases} 0, & \text{if } 1 \leq i,j \leq n \text{ or } n < i,j \leq 2n,\\ q, & \text{if } i = j-n \text{ or } i-1-n = j,\\ -(1+q), & \text{if } i+1 = j-n \text{ or } i = j+1-n,\\ -1, & \text{if } |i-j-n| \geq 2 \text{ and } i > n,\\ 1, & \text{otherwise.} \end{cases} \tag{7}$$

The most general case would have $\nu_{ij} \neq 0$ in the last line of (7). For example, when $n = 4$, then

$$B = \begin{pmatrix} 0 & 0 & 0 & 0 & q & -1-q & 1 & 1\\ 0 & 0 & 0 & 0 & -1-q & q & -1-q & 1\\ 0 & 0 & 0 & 0 & 1 & -1-q & q & -1-q\\ 0 & 0 & 0 & 0 & 1 & 1 & -1-q & q\\ 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0\\ q & 1 & 1 & -1 & 0 & 0 & 0 & 0\\ -1 & q & 1 & 1 & 0 & 0 & 0 & 0\\ -1 & -1 & q & 1 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{8}$$

The bilinear form B in (8) is our particular choice that guarantees that the following equations hold:

$$\begin{aligned} \rho(t_i) &= b_i := \mathsf{e}_i \wedge \mathsf{e}_{i+n}, && (9)\\ b_i b_j &= b_j b_i, \quad \text{whenever} \quad |i-j| \geq 2, && (10)\\ b_i b_{i+1} b_i &= b_{i+1} b_i b_{i+1}. && (11) \end{aligned}$$

This shows ρ to be a homomorphism of algebras implementing the Hecke algebra structure in the Clifford algebra $C\ell(B,V)$. One knows from [11] that ρ is not injective, and that its kernel contains all Young diagrams which are not L-shaped (that is, diagrams with at most one row and/or one column). The first instance, however, when this kernel is non-trivial occurs in S_4 where the partition $4 = (2,2)$ gives a Young diagram of square form which is not L-shaped.

2 The case of $\mathsf{H}_\mathsf{F}(2,\mathsf{q})$ and S_2

We begin with $H_\mathsf{F}(2,q)$ which reduces to S_2 in the limit $q \to 1$. $H_\mathsf{F}(2,q)$ is generated by $\{\mathsf{1}, b_1\}$. We have thus only one q-transposition, from which we can calculate a q-symmetrizer $R(12)$ and a q-antisymmetrizer $C(12)$.

Notice that in the limit $q \to 1$ we have the following relations for a set of new generators defined as $s_i := \lim b_i$ when $q \to 1$:

(i) $s_i^2 = 1$,

(ii) $s_i s_j = s_j s_i$, whenever $|i-j| \geq 2$,

(iii) $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$,

(iii)' $(s_i s_{i+1})^3 = \mathsf{1}$.

Property (iii)' follows from the fact that $s_i^2 = 1$ and $s_i^{-1} = s_i$. This is a presentation of the symmetric group according to Coxeter-Moser [7]. Now it is an easy matter to show that (ii) is valid for transpositions, and that (iii) can be calculated graphically using tangles as in Figure 1. In the cycle notation (iii) can be written as: $(12)(23)(12) =$

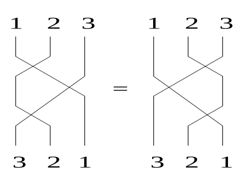

Figure 1: Tangles representing equation (11).

$(23)(12)(23) = (13)$. In the Hecke case it does matter if one 'twists' the tangles one thread over or under the next one, which makes this relation quite nontrivial. It is sometimes called "the quantum-Yang-Baxter equation" in physics. In the case of the symmetric group, it does *not* matter which twist is used. In the crossings of the Hecke algebra we define the left-to-right moving tangle when reading from the top to the bottom as the *upper* one.

Thus, we define the q-symmetrizer $R(12)$ and the q-antisymmetrizer $C(12)$ (up to the normalization) as follows:

$$\begin{aligned} R(12) &:= q + b_1 = q + \mathsf{e}_1 \wedge \mathsf{e}_5, && (12)\\ C(12) &:= \mathsf{1} - b_1 = \mathsf{1} - \mathsf{e}_1 \wedge \mathsf{e}_5. && (13) \end{aligned}$$

Notice, that the q-antisymmetrizer is related to the q-symmetrizer by the operation of reversion denoted by tilde in the Clifford algebra $C\ell_{1,1}$, that is, $C(12)\tilde{} = R(12)$ and $R(12)\tilde{} = C(12)$. How do we know that $q + b_1$ gives the symmetrizer $R(12)$? Notice first that $R(12)$ is almost an idempotent since

$$R(12)R(12) = (1+q)\mathsf{e}_1 \wedge \mathsf{e}_5 + q(1+q) = (1+q)R(12). \tag{14}$$

Thus, when we normalize $R(12)$ by dividing it by $1+q$, the new element denoted as $R(12)_q$ will be an idempotent.

$$R(12)_q R(12)_q = \frac{(1+q)\mathsf{e}_1 \wedge \mathsf{e}_5 + q(1+q)}{(1+q)^2} = \frac{\mathsf{e}_1 \wedge \mathsf{e}_5 + q}{1+q} = \frac{b_1 + q}{1+q} = R(12)_q.$$

If we now take the limit of $R(12)_q$ as $q \to 1$, we obtain

$$\lim_{q \to 1} R(12)_q = \frac{1+s_1}{2} \tag{15}$$

with $s_1 = \mathsf{e}_1 \wedge \mathsf{e}_5$ squaring to $\mathsf{1}$ (in the limit $q \to 1$) in agreement with (i) above. Then the expression $\frac{1}{2}(1+s_1)$ acts as a symmetrizer on, for example, functions of two variables. Likewise, the normalized q-antisymmetrizer gives an idempotent element

$$C(12)_q = R(12)_q\tilde{} = \frac{1-b_1}{1+q} \tag{16}$$

which in the limit $q \to 1$ gives the regular S_2 antisymmetrizer

$$\lim_{q \to 1} C(12)_q = \frac{1-s_1}{2}. \tag{17}$$

Notice that $R(12)_q$, $C(12)_q$ and their limits $\frac{1}{2}(1+s_1)$, $\frac{1}{2}(1-s_1)$ are pairwise mutually annihilating primitive idempotents in $C\ell_{1,1}$.

Therefore, following the standard theory of Young operators, we define the q-Young operator $Y_{1,2}^{(2)}$ as equal to the normalized q-symmetrizer $R(12)_q$ while the q-Young operator $Y_{1,2}^{(11)}$ is defined as equal to the normalized q-antisymmetrizer $C(12)_q$. We can conclude that $Y_{1,2}^{(2)}$ and $Y_{1,2}^{(11)}$ are mutually annihilating idempotents in $H_{\mathsf{F}}(2,q) \subset C\ell_{1,1}^{+} \subset C\ell_{1,1}$ and that they decompose the unity

$$\mathsf{1} = Y_{1,2}^{(2)} + Y_{1,2}^{(11)}. \tag{18}$$

In the Clifford algebra $C\ell_{1,1}$ we can construct various primitive idempotents. Since in our construction the Hecke algebra $H_{\mathsf{F}}(2,q)$ is a subalgebra of the even part $C\ell_{1,1}^{+}$ of $C\ell_{1,1}$, the idempotents of the Hecke algebra must be even Clifford elements.

It can be verified easily with CLIFFORD [2, 3] that the only two nontrivial mutually annihilating idempotents in the Hecke algebra $H_{\mathsf{F}}(2,q)$ are the Young operators found above. In this case, the Young operators happen to be the two even primitive idempotents in the Clifford algebra $C\ell_{1,1}$. [2]

With CLIFFORD we have also looked for any intertwining elements in the algebraic span of the two Young operators $Y_{1,2}^{(2)}$ and $Y_{1,2}^{(11)}$, and we have found none, as expected. That is, we have found no non-trivial element T in the algebraic span of the Hecke algebra such that $TY_{1,2}^{(2)} = Y_{1,2}^{(11)}T$. Since the representations are only one-dimensional here, the Garnir element (see definition below) is zero.

[2] The Clifford algebra $C\ell_{1,1}$ is generated here by the 1-vectors e_1 and e_5. That is, we view $C\ell_{1,1}$ as being embedded in the Clifford algebra $C\ell_{4,4}$ of the 8×8 bilinear form B given in (8).

3 The case of $\mathsf{H}_\mathsf{F}(3,\mathsf{q})$ and S_3

The Hecke algebra $H_\mathsf{F}(3,q)$ is spanned by the basis elements $< \mathbf{1}, b_1, b_2, b_{12}, b_{21}, b_{121} >$ which are expressed in terms of Grassmann polyomials as

$$\begin{aligned} b_1 &= \mathsf{e}_1 \wedge \mathsf{e}_5, \quad b_2 = \mathsf{e}_2 \wedge \mathsf{e}_6, \\ b_{12} = b_1 b_2 &= -(1+q)\mathbf{1} + \mathsf{e}_1 \wedge \mathsf{e}_6 - \mathsf{e}_1 \wedge \mathsf{e}_2 \wedge \mathsf{e}_5 \wedge \mathsf{e}_6 + (1+q)\mathsf{e}_2 \wedge \mathsf{e}_5, \\ b_{21} = b_2 b_1 &= -q(1+q)\mathbf{1} + (1+q)\mathsf{e}_1 \wedge \mathsf{e}_6 - \mathsf{e}_1 \wedge \mathsf{e}_2 \wedge \mathsf{e}_5 \wedge \mathsf{e}_6 + q\mathsf{e}_2 \wedge \mathsf{e}_5, \\ b_{121} = b_1 b_2 b_1 &= q\mathsf{e}_1 \wedge \mathsf{e}_5 - (1+2q)\mathbf{1} + q\mathsf{e}_2 \wedge \mathsf{e}_6 + \mathsf{e}_1 \wedge \mathsf{e}_6 \\ &\quad +(-1+q)\mathsf{e}_1 \wedge \mathsf{e}_2 \wedge \mathsf{e}_5 \wedge \mathsf{e}_6 - (-q-1+q^2)\mathsf{e}_2 \wedge \mathsf{e}_5. \end{aligned} \tag{19}$$

We begin by defining our Young symmetrizer $Y^{(3)}_{1,2,3}$ as in King and Wybourne:

$$Y^{(3)}_{1,2,3} := \frac{q^3\mathbf{1} + q^2 b_1 + q^2 b_2 + q b_{12} + q b_{21} + b_{121}}{(1+q+q^2)(1+q)}. \tag{20}$$

By construction, our Young antisymmetrizer $Y^{(111)}_{1,2,3}$ is defined as the reversion of the symmetrizer $Y^{(3)}_{1,2,3}$, that is,

$$Y^{(111)}_{1,2,3} := Y^{(3)}_{1,2,3}\tilde{} = \frac{\mathbf{1} - b_1 - b_2 + b_{12} + b_{21} - b_{121}}{(1+q+q^2)(1+q)}. \tag{21}$$

Also in this case King and Wybourne's full antisymmetrizer is the reversion of their full symmetrizer. However, the KW Young operators corresponding to Young tableaux which are conjugate of each other in the sense of Mcdonald [27] (see also [19]) and generate representations of dimensions greater than one appear not to be related by the reversion. For example, King and Wybourne define

$$R(13) := q^3\mathbf{1} + b_{121}, \quad C(13) := \mathbf{1} - b_{121} \tag{22}$$

yet $C(13) \neq R(13)\tilde{}$ since

$$\begin{aligned} C(13) - R(13)\tilde{} = 2(q^2-q)\mathbf{1} + (-1-q^2+2q)b_1 + (-1-q^2+2q)b_2 \\ +(1-q)b_{12} + (1-q)b_{21}, \end{aligned} \tag{23}$$

which does equal 0 only in the limit $q \to 1$, that is, not for an arbitrary q. Therefore, the difference between the KW's antisymmetrizer $C(13)$ and their reversed symmetrizer $R(13)\tilde{}$ is the nonzero Hecke element (23) which vanishes automatically in the limit $q \to 1$. This is the reason for us to depart now from the KW formalism and introduce our own definitions of row-symmetrizers and column-antisymmetrizers, and require that they be related through the reversion. By doing so, we realize the conjugation of all Young tableaux as the Clifford algebra reversion. This is a crucial fact: since the reversion is the distinguished anti-automorphism of the Clifford algebra, we are able to connect it here to the automorphism of the Hecke algebra which interchanges the roots in the quadratic Hecke relation $(q,-1) \stackrel{\sim}{\to} (-1,q)$. Such interchange is an element of the Galois group [33]. We have thus established a direct connection between Clifford anti-involutions and the Galois group elements acting in the Hecke algebra. We consider our setting therefore to be in some sense natural.

To be as general as possible, we do the construction of the Young operators in two steps. First, we split the unit element $\mathbf{1}$ in the Hecke algebra using the reversion into two non-primitive idempotents. Each of these idempotents generates a three-dimensional decomposable ideal. To achieve this goal, we can use any two of the following three equations since any two of them imply the third:

$$X + X^{\tilde{}} = \mathbf{1}, \tag{24}$$
$$X^2 = X, \tag{25}$$
$$XX^{\tilde{}} = 0. \tag{26}$$

By doing so, our goal is to find four Young operators known to exist from the general theory of the Hecke algebras for $n = 3$ [20, 34]. The four q-Young operators will still have only one parameter and they will generalize four Young operators of S_3 described in Hamermesh [21] on p. 245. One of them will be a full symmetrizer, another one will be a full antisymmetrizer, and the other two will be of mixed symmetry.

In the first step, we will find the most general element

$$X = K_1\mathbf{1} + K_2 b_1 + K_3 b_2 + K_4 b_{12} + K_5 b_{21} + K_6 b_{121} \tag{27}$$

in the Hecke algebra $H_{\mathsf{F}}(3, q)$ that satisfies (24). Upon substituting X into (24) we have found that X must have the following form:

$$\begin{aligned} X \;=\; & (\tfrac{1}{2}qK_2 - \tfrac{1}{2}qK_6 + \tfrac{1}{2} + \tfrac{1}{2}qK_3 - \tfrac{1}{2}K_2 + \tfrac{1}{2}q^2K_6 - \tfrac{1}{2}K_3)\mathbf{1} \\ & + K_2 b_1 + K_3 b_2 + K_4 b_{12} + (-K_4 + qK_6 - K_6)b_{21} + K_6 b_{121}. \end{aligned} \tag{28}$$

The element X in (28) belongs to a family parameterized by four real or complex parameters K_2, K_3, K_4, K_6. Next we demand that X also satisfies (25).

After substituting X displayed in (28) into equation (25), we have found six sets of solutions. The solutions are parameterized by complex numbers satisfying two similar but different quadratic equations:

$$\begin{aligned} (1+q)z^2 + (-q^2K_4 + K_4 + qK_2 - 1 + K_2)z + K_4K_2 + K_2^2 - K_4 - K_2 \\ + qK_4 - q^2K_4^2 - qK_4^2 - q^2K_2K_4 + qK_2^2 = 0, \end{aligned} \tag{29}$$

$$\begin{aligned} (1+q)z^2 + (-q^2K_4 + K_4 + qK_2 + 1 + K_2)z + K_4K_2 + K_2^2 + K_4 + K_2 \\ - qK_4 - q^2K_4^2 - qK_4^2 - q^2K_2K_4 + qK_2^2 = 0. \end{aligned} \tag{30}$$

Let α be a root of equation (29) and κ be a root of equation (30). We obtain the following six representatives $r_i, i = 1, \ldots, 6,$ of all solution families of (25) which we again express in the Hecke algebra basis:[3]

$$\begin{aligned} r_1 \;=\; & \frac{\mathbf{1}}{1+q} - K_4 b_1 + qK_4 b_2 + K_4 b_{12} - \frac{(q^3K_4 + q + K_4 - 1)b_{21}}{q(1+q)} \\ & - \frac{(-K_4 + q^2K_4 + 1)b_{121}}{q(1+q)}, \end{aligned}$$

$$\begin{aligned} r_2 \;=\; & \frac{q\mathbf{1}}{1+q} - K_4 b_1 + qK_4 b_2 + K_4 b_{12} - \frac{(q^3K_4 - q + K_4 + 1)b_{21}}{q(1+q)} \\ & - \frac{(-K_4 + q^2K_4 - 1)b_{121}}{q(1+q)}, \end{aligned}$$

[3] One might notice, that the roots α and κ provide us two possibly complex solutions each, which yields 8 real solutions: 4 are linearly independent and 4 are related by the reversion.

$$
\begin{aligned}
r_3 &= \frac{\mathbf{1}}{1+q} + qK_4b_1 - K_4b_2 + K_4b_{12} - \frac{(q^3K_4+q+K_4-1)b_{21}}{q(1+q)} \\
&\quad - \frac{(-K_4+q^2K_4+1)b_{121}}{q(1+q)},
\end{aligned}
$$

$$
\begin{aligned}
r_4 &= \frac{q\mathbf{1}}{1+q} + qK_4b_1 - K_4b_2 + K_4b_{12} - \frac{(q^3K_4-q+K_4+1)b_{21}}{q(1+q)} \\
&\quad - \frac{(-K_4+q^2K_4-1)b_{121}}{q(1+q)},
\end{aligned}
$$

$$
\begin{aligned}
r_5 &= \frac{\mathbf{1}}{1+q} + K_2b_1 + K_4b_{12} \\
&\quad - \frac{(\kappa+K_2-qK_4+K_4-q^2K_4^2+qK_2^2-q^2K_2K_4-qK_4^2+K_2^2+K_4K_2)b_2}{(K_2+K_4-qK_4+\kappa)(1+q)} \\
&\quad - \frac{(q\kappa K_4+qK_2K_4+q\kappa K_2-\kappa K_2)b_{21}}{q(K_2+K_4-qK_4+\kappa)} + \frac{(\kappa K_2+qK_4^2)b_{121}}{q(-K_2-K_4+qK_4-\kappa)},
\end{aligned}
$$

$$
\begin{aligned}
r_6 &= \frac{q\mathbf{1}}{1+q} + K_2b_1 + K_4b_{12} \\
&\quad - \frac{(q^2K_4^2-qK_2^2+q^2K_2K_4+qK_4^2+\alpha-qK_4+K_2+K_4-K_2^2-K_4K_2)b_2}{(-K_2-K_4+qK_4-\alpha)(1+q)} \\
&\quad + \frac{(qK_2K_4+q\alpha K_4+q\alpha K_2-\alpha K_2)b_{21}}{(-K_2-K_4+qK_4-\alpha)q} + \frac{(\alpha K_2+qK_4^2)b_{121}}{q(-K_2-K_4+qK_4-\alpha)}.
\end{aligned}
$$

It can be checked with CLIFFORD that the rank of the set $\{r_i\}$, $i=1,\ldots,6$, is four. For our purpose we must select any four linearly independent elements, for example $\{r_1,r_2,r_3,r_5\}$, which we rename $\{f_1,f_2,f_3,f_4\}$. It can also be verified with CLIFFORD that the elements $\{f_1,f_2,f_3,f_4\}$ satisfy the required relations (24), (25), and (26).

We look for the Young operators obtained by one of the $f_i, i=1,\ldots,4$. Due to the fact that the representation spaces which correspond to the symmetric $Y_{1,2,3}^{(3)}$ and the antisymmetric $Y_{1,2,3}^{(111)}$ Young operators respectively are one-dimensional, they cannot have any free parameters besides q. The full symmetrizer can be given according to KW as the q-weighted sum of all six Hecke basis elements. However, in our construction the full antisymmetrizer is defined as the reversion of the full symmetrizer, that is, $Y_{1,2,3}^{(111)} := Y_{1,2,3}^{(3)}\tilde{}$, as it was done in dimension two. Then we have:

$$
Y_{1,2,3}^{(3)} := \frac{q^3\mathbf{1}+q^2b_1+q^2b_2+qb_{12}+qb_{21}+b_{121}}{(1+q+q^2)(1+q)}, \tag{31}
$$

$$
Y_{1,2,3}^{(111)} := \frac{\mathbf{1}-b_1-b_2+b_{12}+b_{21}-b_{121}}{(1+q+q^2)(1+q)}. \tag{32}
$$

Each of the four f_i elements defined above generates a three-dimensional one-sided ideal in the Hecke algebra, or, in other words, together with its reversion $\tilde{f}_i$ it decomposes the unity in $H_{\mathsf{F}}(3,q)$. Our objective is to further split each of these ideals into one- and two-dimensional vector spaces. The one-dimensional vector spaces will be generated by the full symmetrizer (31) and the full antisymmetrizer (32) respectively. The two-dimensional vector spaces will be generated by the mixed type Young operators. So each of the f_i elements has to be a sum of a full (anti)symmetrizer and a Young operator of the mixed type. If we pick f_1 we notice that it must contain the full antisymmetrizer, because when the parameter K_4 is replaced with $1/(q+1)$ then the re-defined f_1 (or the r_1 defined above) reduces to an expression with alternating signs in the Hecke basis:

$$\frac{\mathsf{1}}{1+q} - \frac{b_1}{1+q} + \frac{qb_2}{1+q} + \frac{b_{12}}{1+q} - \frac{qb_{21}}{1+q} - \frac{b_{121}}{1+q}.$$

Therefore, by subtracting the full antisymmetrizer $Y^{(111)}_{1,2,3}$ from f_1 we find our first Young operator $Y^{(21)}_{1,3,2}$ of the mixed type:

$$\begin{aligned} Y^{(21)}_{1,3,2} &= f_1 - Y^{(111)}_{1,2,3} \\ &= \frac{q\mathsf{1}}{q+1+q^2} - \frac{(q^3K_4+2q^2K_4+2qK_4-1+K_4)b1}{q^3+2q^2+2q+1} \\ &\quad + \frac{(K_4q^4+2q^3K_4+2q^2K_4+qK_4+1)b_2}{(q^3+2q^2+2q+1)} \\ &\quad + \frac{(q^3K_4+2q^2K_4+2qK_4-1+K_4)b_{12}}{(q^3+2q^2+2q+1)} \\ &\quad - \frac{(K_4q^5+K_4q^4+q^3K_4+q^3+q^2K_4+qK_4+q+K_4-1)b_{21}}{(q^3+2q^2+2q+1)q} \\ &\quad - \frac{(K_4q^4+q^3K_4+q^2-qK_4+1-K_4)b_{121}}{(q+1+q^2)q(1+q)}. \end{aligned} \tag{33}$$

We define our second Young operator $Y^{(21)}_{1,2,3}$ of the mixed type as the reversion (conjugate) of $Y^{(21)}_{1,3,2}$, that is, $Y^{(21)}_{1,2,3} := Y^{(21)\,\tilde{}}_{1,3,2}$, and we get:

$$\begin{aligned} Y^{(21)}_{1,2,3} &= \frac{q\mathsf{1}}{q+1+q^2} + \frac{(q^3K_4-q^2+2q^2K_4+2qK_4+K_4)b_1}{(1+q)(q+1+q^2)} \\ &\quad - \frac{q(q^3K_4+2q^2K_4+2qK_4+q+K_4)b_2}{(1+q)(q+1+q^2)} \\ &\quad - \frac{(q^3K_4+2q^2K_4+2qK_4+q+K_4)b_{12}}{q^3+2q^2+2q+1} \\ &\quad + \frac{(K_4q^5+K_4q^4+q^3K_4+q^3-q^2+q^2K_4+qK_4-1+K_4)b_{21}}{q(q^3+2q^2+2q+1)} \\ &\quad + \frac{(K_4q^4+q^3K_4+q^2-qK_4+1-K_4)b_{121}}{(q+1+q^2)q(1+q)}. \end{aligned} \tag{34}$$

Furthermore, the idempotent $Y^{(111)}_{1,2,3}$ (resp. $Y^{(3)}_{1,2,3}$) annihilates the idempotent $Y^{(21)}_{1,3,2}$ (resp. $Y^{(21)}_{1,2,3}$) when multiplied from both sides. This reflects the fact, that the representation spaces constructed in this way give a direct sum decomposition of the three-dimensional ideal generated by f_1 (resp. $\tilde{f}_1$).

Let's summarize the relationships between the Young operators:

$$\begin{gathered} f_1 = Y^{(111)}_{1,2,3} + Y^{(21)}_{1,3,2} \quad , \quad Y^{(21)}_{1,2,3} + Y^{(3)}_{1,2,3} = \tilde{f}_1, \\ Y^{(111)}_{1,2,3}Y^{(21)}_{1,3,2} = Y^{(21)}_{1,3,2}Y^{(111)}_{1,2,3} = 0 \quad , \quad Y^{(21)}_{1,2,3}Y^{(3)}_{1,2,3} = Y^{(3)}_{1,2,3}Y^{(21)}_{1,2,3} = 0, \\ Y^{(111)}_{1,2,3} = Y^{(3)\,\tilde{}}_{1,2,3} \quad , \quad Y^{(21)}_{1,3,2} = Y^{(21)\,\tilde{}}_{1,2,3}. \end{gathered} \tag{35}$$

Thus, we have carefully built in the conjugation of the Young tableaux as the Clifford reversion. It can then be checked with CLIFFORD that the four Young operators displayed in (31), (32), (33), and (34) are primitive mutually annihilating idempotents which decompose the unity in the Hecke algebra since $f_1 + \tilde{f}_1 = \mathbf{1}$. It can be easily verified that the Young operators of mixed type decompose into the row-symmetrizer and the colum-antisymmetrizer in accordance to Hamermesh [21] p. 245. Our expressions however are different from those in KW.

In order to represent our Young operator $Y^{(21)}_{1,3,2}$ as a product of the row symmetrizer $R(13)$ and the column antisymmetrizer $C(12)$, we use previously defined $f_1 = r_1$ to define $C(12) := f_1$ and compute $R(13)$ from the equation

$$Y^{(21)}_{1,3,2} = R(13) f_1. \tag{36}$$

Notice that our $f_1 = r_1$ is a generalization to S_3 of $C(12)$ from S_2 displayed in (13). In an effort to be consistent with our previous discussion of $C(12)$ and $R(12)$, which were related by the reversion, we will later define $C(13) := R(13)\tilde{}$ and require that $R(13) + C(13) = R(13) + R(13)\tilde{} = \mathbf{1}$. Thus, when we solve (36) for $R(13)$, we get the following solution:

$$\begin{aligned} R(13) \;=\; & \frac{q\mathbf{1}}{1+q} - \frac{(-q^2 + q^2 P_3 + P_3 q - 1 + P_3) b_1}{(q+1+q^2)q} + P_3 b_2 \\ & + \frac{(q^2 P_3 + P_3 q + P_3 - 1) b_{12}}{q(1+q+q^2)} \\ & - \frac{(q^5 P_3 + q^4 P_3 + q^3 P_3 + q^2 P_3 - q^2 + P_3 q - 1 + P_3) b_{21}}{(1+q)q^2(1+q+q^2)} \\ & - \frac{(q^4 P_3 + q^3 P_3 - P_3 q + 1 - P_3) b_{121}}{(1+q)q^2(1+q+q^2)} \end{aligned} \tag{37}$$

where P_3 is an arbitrary real or complex parameter. It can be also verified with CLIFFORD that $R(13)$ is an idempotent element in the Hecke algebra which is not a feature in KW. It is an interesting fact to note that the free parameter P_3 disappears when the product $R(13) f_1 = Y^{(21)}_{1,3,2}$ is taken: the only free remaining parameter is K_4. As a natural consequence of (36) we have

$$Y^{(21)}_{1,2,3} = \tilde{f}_1 R(13)\tilde{}. \tag{38}$$

Obviously, we also have the following relations:

$$C(13) := R(13)\tilde{}, \quad C(12) := f_1, \quad R(12) := C(12)\tilde{}. \tag{39}$$

There are no non-trivially intertwining elements in the Hecke algebra which would connect the four Young operators except for the pair $\{Y^{(21)}_{1,2,3}, Y^{(21)}_{1,3,2}\}$. That is, the only intertwiners that connect $Y^{(3)}_{1,2,3}$ with $Y^{(111)}_{1,2,3}$ actually annihilate them, and similarly for the pairs $\{Y^{(3)}_{1,2,3}, Y^{(21)}_{1,2,3}\}$ and $\{Y^{(111)}_{1,2,3}, Y^{(21)}_{1,3,2}\}$. On the other hand, there are many choices for an element T in the Hecke algebra such that

$$T Y^{(21)}_{1,2,3} = Y^{(21)}_{1,3,2} T \neq 0. \tag{40}$$

In fact, T belongs to a five-parameter family of solutions. By assigning 0 and 1 values to four of those parameters, we have reduced the solutions to a one-parameter family parameterized only by K_4 as follows:

$$\begin{aligned} T &= \frac{(1-K_4-q^3-K_4q+K_4q^3+q^4K_4)\mathsf{1}}{q(2K_4-q^2-q-q^3+2q^4K_4+8K_4q^2+6K_4q+6K_4q^3-1)} \\ &\quad+\frac{2(-1-q-q^2+K_4+2K_4q+K_4q^3+2K_4q^2)b_1}{q(2K_4-q^2-q-q^3+2q^4K_4+8K_4q^2+6K_4q+6K_4q^3-1)} \\ &\quad-\frac{b_2}{1+q}-\frac{b_{12}}{q(1+q)}+\frac{b_{21}}{1+q}+\frac{b_{121}}{q(1+q)} \end{aligned} \tag{41}$$

where $K_4 \neq \dfrac{(q^2+1)}{2(q^3+2q^2+2q+1)}$. Thus, the only intertwiners in the Hecke algebra which are not annihilators and which connect the mixed type Young operators constitute, in general, a four-parameter family of intertwiners while T gives a one-parameter family.

We introduce now the Garnir elements $G_{i,j}^{(\lambda_i)}$ (see KW) which will allow us to construct the six-dimensional representation spaces from the Young idempotents. Garnir elements can be seen to act as row or column permutations in Young tableaux, thereby generating non-standard tableaux which correspond to the basis vectors of the representation space. A *Garnir element* $G_{1,1}^{(21)}$ has the following defining properties:

$$Y_{1,2,3}^{(21)}\,G_{1,1}^{(21)} = 0, \tag{42}$$

$$G_{1,1}^{(21)}\,Y_{1,2,3}^{(21)} \neq 0. \tag{43}$$

Our goal is to use a suitable Garnir element to decompose the three-dimensional one sided ideals generated by the Young operators of mixed symmetry $Y_{1,3,2}^{(21)}$ and $Y_{1,2,3}^{(21)}$ (see (33) and (34) respectively) into a direct sum of one-dimensional and two-dimensional vector spaces.

When we first require that a general Hecke element X shown in (27) satisfies equation (42) when substituted for $G_{1,1}^{(21)}$, we find three different linearly independent solutions which we call XX_1, XX_2, and XX_3. In the Hecke basis they look as follows:

$$XX_1 = (-K_6q+K_2q+K_4)\mathsf{1}+K_2b_1+\frac{t_1b_2}{q}+K_4b_{12}+K_5b_{21}+K_6b_{121}, \tag{44}$$

$$XX_2 = K_1\mathsf{1}+\frac{t_2b_1}{q^2(1+q)}+\frac{t_3b_2}{q^3(1+q)}-\frac{b_{12}}{q(1+q)}+K_5b_{21}+K_6b_{121}, \tag{45}$$

$$\begin{aligned} XX_3 &= K_1\mathsf{1}+\frac{t_4b_1}{q(q^3+2q^2+2q+1)}+\frac{t_5b_2}{q^2(q^3+2q^2+2q+1)} \\ &\quad-\frac{(q-1)b_{12}}{q^3+2q^2+2q+1}+K_5b_{21}+K_6b_{121}, \end{aligned} \tag{46}$$

where the polynomial t_1 is parameterized by K_2, K_4, K_5, K_6, and the polynomials t_2, t_3, t_4, t_5 are parameterized by K_1, K_5, K_6.[4]

[4] $t_1 = K_6q^2+K_4q^2-K_5q-K_4q-K_6q+K_2+K_4$, $t_2 = K_6q^3+K_6q^2+q^2K_1+qK_1+1$, $t_3 = q^5K_6-q^4K_5-q^3-q^3K_5+q^2K_1+K_6q^2+q^2+qK_1-q+1$, $t_4 = q-1+K_6q+2K_6q^2+2K_6q^3+2qK_1+2q^2K_1+K_1+q^3K_1+q^4K_6$, $t_5 = q^6K_6+q^5K_6-q^5K_5+q^4K_6-q^4-2q^4K_5+K_6q^3+q^3K_1+2q^3-2q^3K_5-K_5q^2+K_6q^2+2q^2K_1-2q^2+2qK_1+K_6q+2q+K_1-1$.

With CLIFFORD it has been verified that the equation (43) is satisfied automatically by each $XX_i, i = 1, 2, 3,$ for all possible choices of the parameters. This means that we can use any of the three solutions as the Garnir element. Our choice is:

$$G_{1,1}^{(21)} \;=\; (-K_6 q + K_2 q + K_4)\mathbf{1} + K_2 b_1 + \frac{t_1 b_2}{q} + K_4 b_{12} + K_5 b_{21} + K_6 b_{121}. \tag{47}$$

Next we introduce the q-automorphism α_q which replaces the reversion in changing the Garnir elements when acting from the left. This automorphism is in fact the inverse of the b_i's generators and their products, the Hecke versors, and it is linearly extended to the entire Hecke algebra by means of the following definition:

$$\begin{aligned} \alpha_q(b_{i_1}\ldots b_{i_s}) &= (\tfrac{-1}{q})^s (b_{i_1}\cdots b_{i_s})^{\tilde{}} \\ &= (\tfrac{-1}{q})^s (\tilde{b_{i_s}}\cdots \tilde{b_{i_1}}) \\ &= \alpha_q(b_{i_s})\cdots\alpha_q(b_{i_1}). \end{aligned} \tag{48}$$

For example,

$$\alpha_q(b_1) = \frac{(q-1)\mathbf{1}}{q} + \frac{b_1}{q}, \quad \alpha_q(b_2) = \frac{(q-1)\mathbf{1}}{q} + \frac{b_2}{q},$$
$$\alpha_q(b_1 b_2) = \frac{(1-2q+q^2)\mathbf{1}}{q^2} + \frac{(q-1)b_1}{q^2} + \frac{(q-1)b_2}{q^2} + \frac{b_{21}}{q^2}.$$

Note the fact that

$$\begin{aligned} \alpha_q(b_i)b_i &= \tfrac{-1}{q}(\tilde{b_i} b_i) \\ &= \tfrac{-1}{q}(((1-q)-b_i)b_i) \\ &= \tfrac{-1}{q}((1-q)b_i - (1-q)b_i - q) \;=\; \mathbf{1}. \end{aligned} \tag{49}$$

In fact, α_q acts as the inverse on the generators and, as such, it depends on the presentation. However, while α_q can be extended by the linearity to the entire Hecke algebra, it gives the inverse only of the products of the b_i's (versors) and not of their sums. For example,

$$\alpha_q(\mathbf{1}+b_1) = \frac{(2q-1)\mathbf{1}}{q} + \frac{b_1}{q} \;\neq\; (\mathbf{1}+b_1)^{-1} = \frac{(-2+q)\mathbf{1}}{2(q-1)} + \frac{b_1}{2(q-1)},\; q \neq 1. \tag{50}$$

One can check that it is not possible to use the reversion to connect the left and the right actions of Garnir elements even if this transformation connects the Young operators of conjugated diagrams. Looking at α_q as an q-inverse, one could try to form a q-Clifford-Lipschitz group [15] in the Hecke algebra by defining:

$$\Gamma_q := \{X \mid \alpha_q(X)X = \mathbf{1}\}. \tag{51}$$

When the automorphism α_q is applied to the Garnir element $G_{1,1}^{(21)}$, one gets

$$\begin{aligned} \alpha_q(G_{1,1}^{(21)}) \;=\; & -\frac{t_6\mathbf{1}}{q^3} + \frac{t_7 b_1}{q^3} - \frac{t_8 b_2}{q^3} + \frac{(-K_6 + K_6 q + K_5 q)b_{12}}{q^3} \\ & + \frac{(-K_6 + K_4 q + K_6 q)b_{21}}{q^3} + \frac{K_6 b_{121}}{q^3}, \end{aligned} \tag{52}$$

where t_6, t_7, t_8 are polynomials.[5] With CLIFFORD we have verified that the six elements in the list S below are linearly independent. As such, they provide a basis for the left regular representation of the Hecke algebra:

$$S = [Y^{(3)}_{1,2,3},\ Y^{(21)}_{1,2,3},\ G^{(21)}_{1,1}Y^{(21)}_{1,2,3},\ \alpha_q(G^{(21)}_{1,1})Y^{(21)}_{1,3,2},\ Y^{(21)}_{1,3,2},\ Y^{(111)}_{1,2,3}]. \tag{53}$$

Recall that the original basis in the Hecke algebra was $[\mathbf{1}, b_1, b_2, b_{12}, b_{21}, b_{121}]$. Each original basis element should be representable in terms of the new basis S. Then we have:

$$\mathbf{1} = S_1 + S_2 + S_5 + S_6,$$

$$\begin{aligned} b_1 &= S_1 + \frac{(K_4q^3 + 2K_4q^2 - q^2 + 2K_4q + K_4)S_2}{1+q} \\ &\quad + \frac{p_1qS_3}{p_2(1+q)} + \frac{q^2S_4}{p_3} + \frac{q(-K_4q - K_2q + K_6 + K_5)S_5}{p_4} - qS_6, \end{aligned}$$

$$b_2 = S_1 - \frac{qp_5S_2}{1+q} - \frac{p_6qS_3}{p_2(1+q)} - \frac{q^3S_4}{p_3} - \frac{p_7S_5}{p_4} - qS_6,$$

$$b_{12} = S_1 + \frac{p_8S_2}{1+q} + \frac{p_9qS_3}{p_2(1+q)} + \frac{(1-q+q^2)q^2S_4}{p_3} - \frac{qp_{10}S_5}{p_4} + q^2S_6,$$

$$b_{21} = S_1 - \frac{qp_5S_2}{1+q} - \frac{p_{11}q^2S_3}{p_2(1+q)} - \frac{q^3S_4}{p_3} - \frac{q^2(-K_4q - K_2q + K_6 + K_5)S_5}{p_4} + q^2S_6,$$

$$b_{121} = S_1 + \frac{p_{13}qS_2}{1+q} + \frac{p_{1,2}q^2S_3}{p_2(1+q)} + \frac{(q-1)q^3S_4}{p_3} - \frac{q^2p_{14}S_5}{p_4} - q^3S_6$$

where to shorten the display we have introduced polynomials $p_i,\ i = 1, \ldots, 14$ (see the Appendix) and S_i denotes the i-th element of the list $S,\ i = 1, \ldots, 6$. Finally, we can find, as expected, block-structured matrices of the basis elements b_1 and b_2 in the left regular representation (additional polynomials $p_i,\ i = 15, \ldots, 19$, are also given in the Appendix). They are as follows:

$$M_{b_1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \dfrac{p_{15}}{1+q} & -\dfrac{p_{16}}{q(1+q)} & 0 & 0 & 0 \\ 0 & \dfrac{qp_1}{p_{17}} & -\dfrac{p_{18}}{1+q} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\dfrac{p_{19}}{p_4} & \dfrac{q^2}{p_3} & 0 \\ 0 & 0 & 0 & -\dfrac{p_{20}}{q\,p_4} & \dfrac{qp_{21}}{p_4} & 0 \\ 0 & 0 & 0 & 0 & 0 & -q \end{pmatrix} \tag{54}$$

[5] $t_6 = K_2q - 2K_6q + K_6q^3 + K_6q^2 + K_5q^2 + K_6 - q^3K_2 - q^4K_2 - K_5q - q^4K_4$, $t_7 = K_6 + q^2K_2 - K_5q + K_5q^2 - 2K_6q + K_4q^2 - K_4q + K_6q^2$, $t_8 = -K_2q + 2K_6q - K_6 + K_5q - K_6q^3 - K_4q^3$.

$$M_{b_2} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\dfrac{qp_5}{1+q} & \dfrac{p_{16}}{1+q} & 0 & 0 & 0 \\ 0 & -\dfrac{qp_6}{p_{17}} & \dfrac{p_{21}}{1+q} & 0 & 0 & 0 \\ 0 & 0 & 0 & \dfrac{qp_{22}}{p_4} & \dfrac{-q^3}{p_3} & 0 \\ 0 & 0 & 0 & -\dfrac{p_{23}}{qp_4} & -\dfrac{p_7}{p_4} & 0 \\ 0 & 0 & 0 & 0 & 0 & -q \end{pmatrix} \tag{55}$$

Matrices M_{b_1} and M_{b_2} satisfy, of course, the same quadratic Hecke relation (3) as do b_1 and b_2, and which happens to give the minimum polynomial $p(x) = x^2 - (1-q)x - q = (x-1)(x+q)$ for the basis elements and their matrix representations. The characteristic polynomial for the latter is $c(x) = (x-1)^3(x+q)^3$. The trace of M_{b_1} and M_{b_2} is $3(q-1)$ while their determinants equal $-q^3$.

Finally, we build a left-regular matrix representation of the Young basis elements (53) in the Young basis.

$$M_{S_1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad M_{S_2} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \dfrac{p_{24}}{q^2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{56}$$

$$M_{S_3} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\dfrac{p_{25}}{q^3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad M_{S_4} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\dfrac{p_{25}}{q^3} & 0 & 0 & 0 \\ 0 & 0 & -\dfrac{p_{24}}{q^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\dfrac{p_{24}}{q^2} & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{57}$$

$$M_{S_5} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\dfrac{p_{24}}{q^2} & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad M_{S_6} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{58}$$

4 Conclusions

Motivated by the desire to describe symmetries of q-multiparticle systems we have constructed q-symmetrizers and q-antisymmetrizers in the Hecke algebras $H_{\mathsf{F}}(2,q)$ and $H_{\mathsf{F}}(3,q)$, and have related them by the reversion. That is, the Young operators constructed in the paper corresponding to the Young tableaux conjugate to each other in the sense of Macdonald and which generate representation spaces of dimension greater than one, have been related through the reversion in the Clifford algebra $C\ell_{4,4}$. This feature is not present in King and Wybourne.

In $H_{\mathsf{F}}(2,q)$ we found that the symmetrizer $R(12)$ and its reverse, the antisymmetrizer $C(12)$ were primitive idempotents in the Clifford algebra. We found no non-trivial intertwiners linking these two idempotents.

In $H_{\mathsf{F}}(3,q)$ we first found four mutually annihilating idempotents splitting the unity in the algebra: two without parameters and two parameterized ones. The first two were the Young symmetrizer $Y^{(3)}_{1,2,3}$, defined as in King and Wybourne, and the Young antisymmetrizer $Y^{(111)}_{1,3,2}$, defined in this paper as the reverse of $Y^{(3)}_{1,2,3}$. The two parameterized idempotents were the Young operators of mixed symmetry $Y^{(21)}_{1,2,3}$ and $Y^{(21)}_{1,3,2}$, and they were also related by the reversion. We were able to factor $Y^{(21)}_{1,3,2}$ into the row symmetrizer $R(13)$ and the column antisymmetrizer $C(12)$, that is, we were able to find $R(13)$ as an idempotent element in the Clifford algebra, a feature not found in King and Wybourne. Furthermore, we related the mixed-type Young operators through a five-parameter family of intertwiners.

We have found a Garnir element $G^{(21)}_{1,1}$ (in fact, three distinct families of such elements) which allowed us to further split the representation space of $H_{\mathsf{F}}(3,q)$ from $3\oplus 3$ to $1\oplus 2\oplus 2\oplus 1:$ the one-dimensional spaces being generated by the Young symmetrizer $Y^{(3)}_{1,2,3}$ and the Young antisymmetrizer $Y^{(111)}_{1,3,2}$ while the two-dimensional spaces being generated by $\{Y^{(21)}_{1,2,3}, G^{(21)}_{1,1}Y^{(21)}_{1,2,3}\}$ and $\{\alpha_q(G^{(21)}_{1,1})Y^{(21)}_{1,3,2}, Y^{(21)}_{1,3,2}\}$ respectively. We introduced a Hecke algebra automorphism α_q which acted as the inverse when applied to the Hecke basis elements: when applied to the Garnir element α_q allowed us to generate the representation space of dimension six. The α_q automorphism is expected to be useful in defining a q-Clifford-Lipschitz group in the Hecke algebra [15]. Finally, we computed the matrix representation of the Hecke generators b_1 and b_2 in the Young basis.

In the next step to be considered elsewhere we intend to extend this approach to $H_{\mathsf{F}}(4,q)$. The connection with the spinor representations of the appropriate Clifford algebras in all three cases needs to be explored. Our approach to the Young operators related to the Young tableaux of various symmetries as Clifford (and Hecke) idempotents has been motivated by the need to describe the q-symmetries of multiparticle states. We have been able to define and construct all needed notations for a classification of tensor spaces w.r.t. the q-symmetry. As mentioned above, the next natural step is to compute multiparticle spinors, that is, spin-tensors, as they are used for mesons or hadrons in QFT. Such spinors are, however, not connected with the spinors of the Clifford algebras used in this paper, but instead they should be connected with spinors of some appropriate sub-Clifford algebras and their tensor product. Our aim to break up a 'container Clifford algebra' into smaller pieces will thus be achieved. We found that the Clifford algebra framework has proven to be most useful for this task.

Appendix

Polynomials below have been introduced as abbreviation to improve readability of the formulas displayed in the main text:

$$
\begin{aligned}
p_1 &= q^4K_4^2+3q^3K_4^2-K_4q^3+4q^2K_4^2-K_4q^2+3qK_4^2 \\
&\quad -K_4q-q+K_4^2-K_4, \\
p_2 &= K_4q^4K_6+q^4K_4^2+q^3K_4K_6-q^3K_4K_2+K_4q^2 \\
&\quad -q^2K_2K_4+q^2K_4K_6+K_6q^2+q^2K_4K_5-q^2K_4^2 \\
&\quad -K_2q-K_6q+qK_4K_5-qK_4^2+K_4qK_6-K_5q \\
&\quad -qK_2K_4-K_4q-K_4^2-K_4K_2+K_4+K_2, \\
p_3 &= -K_6q^3-K_4q^3+K_6q^2+K_5q^2+K_6q+K_5q-K_6-K_4, \\
p_4 &= -K_6q^2-K_4q^2+2K_6q+K_5q+K_4q-K_6-K_4, \\
p_5 &= K_4q^3+2K_4q^2+q+2K_4q+K_4, \\
p_6 &= q^5K_4^2+3q^4K_4^2+4q^3K_4^2+K_4q^3+3q^2K_4^2 \\
&\quad +K_4q^2+qK_4^2+K_4q+K_4-1, \\
p_7 &= -K_4q^3-q^3K_2+2K_6q^2+K_5q^2+K_4q^2-2K_6q-K_5q \\
&\quad -K_4q+K_6+K_4, \\
p_8 &= q^5K_4+q^4K_4+q^3+K_4q^3-q^2+K_4q^2+K_4q+K_4-1, \\
p_9 &= q^6K_4^2+2q^5K_4^2+2q^4K_4^2+2q^4K_4+2q^3K_4^2 \\
&\quad +K_4q^3+2q^2K_4^2+q^2+2qK_4^2-K_4q-q+K_4^2-2K_4+1, \\
p_{10} &= q^3K_2-K_6q^3+K_6q^2-q^2K_2+K_5q+K_2q-K_5-K_6, \\
p_{11} &= q^4K_4^2+3q^3K_4^2+4q^2K_4^2+K_4q^2+3qK_4^2 \\
&\quad +K_4^2-K_4+1, \\
p_{12} &= q^5K_4^2+2q^4K_4^2+q^3K_4^2+2K_4q^3-q^2K_4^2 \\
&\quad +2K_4q^2-2qK_4^2+2K_4q+q-K_4^2+2K_4-1, \\
p_{13} &= q^4K_4+K_4q^3+q^2-K_4q-K_4+1, \\
p_{14} &= q^2K_2-K_6q^2+K_6q-K_2q-K_4+K_5, \\
p_{15} &= K_4q^3+2K_4q^2-q^2+2K_4q+K_4, \\
p_{16} &= 3K_4q^4K_6+3q^3K_4K_6+K_4qK_6+2q^2K_4K_6 \\
&\quad +2q^5K_4K_6+2q^2K_4K_5+K_6q^4-2q^4K_4K_2 \\
&\quad +qK_4K_5+2q^3K_4K_5-q^3K_2-K_5q-K_6q-K_5q^2 \\
&\quad -2qK_2K_4+K_2+q^6K_4K_6+q^4K_4K_5+K_4 \\
&\quad -3q^2K_2K_4+q^4K_4+K_4q^2-K_4^2-3q^3K_4K_2 \\
&\quad -q^5K_4K_2-2q^3K_4^2-q^3K_5+q^6K_4^2+q^5K_4^2 \\
&\quad -3q^2K_4^2-2qK_4^2-K_4K_2, \\
p_{17} &= 2K_4q^4K_6+2q^3K_4K_6+K_4qK_6+2q^2K_4K_6 \\
&\quad +q^5K_4K_6+2q^2K_4K_5-q^4K_4K_2+qK_4K_5 \\
&\quad +q^3K_4K_5-K_5q-K_6q-K_5q^2-2qK_2K_4+K_2 \\
&\quad +q^4K_4^2+K_4-2q^2K_2K_4+K_6q^3-q^2K_2+K_4q^3 \\
&\quad -K_4^2-2q^3K_4K_2-q^3K_4^2+q^5K_4^2-2q^2K_4^2 \\
&\quad -2qK_4^2-K_4K_2,
\end{aligned}
$$

$$
\begin{aligned}
p_{18} &= K_4q^3 + 2K_4q^2 + 2K_4q - 1 + K_4, \\
p_{19} &= -K_4q^3 - K_6q^3 + K_4q^2 + 3K_6q^2 + K_5q^2 - q^2K_2 \\
&\quad -2K_4q - 2K_6q + K_6 + K_4, \\
p_{20} &= K_4q^4K_6 - 3q^3K_4K_6 + 2K_4qK_6 - 2q^2K_4K_6 \\
&\quad -q^5K_4K_6 - q^2K_4K_5 - K_6K_4 + q^4K_4K_2 + qK_4K_5 \\
&\quad -q^3K_4K_5 - qK_2K_4 + q^4K_4^2 + q^2K_2K_4 - K_4^2 \\
&\quad +2K_6^2q - K_6^2q^2 + 2q^3K_4K_2 + K_5K_4 + q^5K_4K_2 \\
&\quad +K_6q^5K_2 - K_5q^4K_2 - qK_2K_6 + K_6K_5 + q^3K_2^2 \\
&\quad -K_5q^2K_2 + K_6K_5q + K_5q^4K_6 + K_5q^3K_6 \\
&\quad +2K_6^2q^4 - K_6^2q^5 - 2q^3K_2K_5 - 2q^4K_2K_6 \\
&\quad -2q^3K_2K_6 + q^4K_2^2, \\
p_{21} &= -K_4q - K_2q + K_6 + K_5, \\
p_{22} &= -q^2K_2 + K_6q^2 - K_6q - K_4q + K_6 + K_4, \\
p_{23} &= K_4q^4K_6 - q^3K_4K_6 + q^5K_4K_6 - q^2K_4K_5 \\
&\quad -K_6K_4 - q^4K_4K_2 - qK_4K_5 - q^3K_4K_5 + qK_2K_4 \\
&\quad +q^2K_2K_4 - K_6^2 - K_6^2q^3 + K_6^2q^2 - K_5K_4 - q^5K_4K_2 \\
&\quad +K_6q^5K_2 + K_5q^4K_2 + qK_2K_6 - K_6K_5 - K_5q^2K_2 \\
&\quad +K_6K_5q + 2K_6q^2K_5 + K_5^2q^2 - q^5K_2^2 + K_5^2q \\
&\quad -K_5q^4K_6 - K_5q^3K_6 - K_6^2q^4 + 2q^4K_2K_6 - q^4K_2^2 \\
&\quad +q^2K_4^2 + qK_4^2, \\
p_{24} &= -q^4K_4^2 + K_6 - q^2K_2 + 2K_6q^2 - q^5K_4K_6 \\
&\quad +2q^3K_4K_5 + qK_4K_5 - q^5K_4^2 - K_6K_4 - K_6q - K_4q^3 \\
&\quad +q^2K_4K_6 + K_4 + K_5q^2 - q^2K_4^2 - q^3K_2 + q^3K_4K_6 \\
&\quad +2q^2K_4K_5 + q^4K_4K_5 - K_4q - qK_4^2 - q^3K_4^2 - K_4^2, \\
p_{25} &= -3K_4q^4K_6 + 3q^3K_4K_6 + 3K_4qK_6 - 3q^2K_4K_6 + 3q^5K_4K_6 \\
&\quad -K_6K_4 - q^4K_4K_2 + 2qK_4K_5 + 2q^3K_4K_5 + qK_2K_4 - 2q^6K_4K_6 \\
&\quad -K_5^2q^4 - K_6^2q^6 - q^3K_5^2 - K_4^3q^8 + q^4K_4^3 + q^5K_4^3 - 2q^4K_4^2 + 2K_6q^5K_5 \\
&\quad +qK_4^3 + 2q^2K_4^3 + q^3K_4^3 + K_4q^4K_6^2 + K_4^2q^7K_5 + K_4^2q^7K_2 - 2K_4^2q^8K_6 \\
&\quad -q^2K_2K_4 + 3q^5K_5K_4K_6 - 2q^5K_2K_4K_5 - q^4K_2K_4K_6 + 3K_5q^3K_4K_6 \\
&\quad -3K_2q^3K_4K_5 + 2K_5^2q^4K_4 + K_6q^7K_4K_5 + q^7K_2K_4K_6 - K_6^2q^8K_4 \\
&\quad -3K_2q^4K_4K_5 - K_2q^3K_4K_6 + 4K_5q^4K_4K_6 + K_6^2q^3 - K_4^2 + 2q^5K_4K_5 \\
&\quad +K_6^2q - 2K_6^2q^2 + q^3K_4K_2 + q^5K_4K_2 + K_5^2q^5K_4 - K_4^2q^6K_6 + 2q^3K_4^2K_2 \\
&\quad +2q^4K_4^2K_2 + 2q^5K_4^2K_2 + q^6K_4^2K_2 - 4q^3K_4^2K_6 - 3K_4^2q^4K_6 - K_4^2q^5K_6 \\
&\quad -3q^4K_4^2K_5 - q^5K_4^2K_5 - q^2K_4^2K_6 + 2q^2K_2K_4^2 - 3q^2K_4^2K_5 - 4q^3K_4^2K_5 \\
&\quad +qK_2K_4^2 + K_4^3 + q^6K_5K_4K_6 + K_6q^5K_2 + q^3K_4^2 - K_5q^4K_2 + 2qK_2K_6 \\
&\quad +K_6K_5q - 2K_6q^2K_5 - K_5^2q^2 - K_5q^4K_6 - K_6^2q^4 + 2K_6^2q^5 - 2q^4K_2K_6 \\
&\quad +q^3K_2K_6 + K_4q^5K_6^2 - q^6K_4^2 + q^5K_4^2 - 2q^2K_5K_4K_2 + q^2K_5K_4K_6 \\
&\quad +K_6K_4K_2 + K_4^2K_2 + K_4^2K_6 - 2qK_4^2K_5 - qK_4^2K_6 - K_6K_2 - qK_6K_4K_5 \\
&\quad -qK_5K_2K_4 + 2K_5^2K_4q^3 + K_5K_2q - q^6K_5K_2K_4 \\
&\quad -K_6K_2q^2 + K_5^2K_4q^2 - K_6^2K_4q - 2q^2K_4^2 + qK_4^2 - K_4K_2.
\end{aligned}
$$

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