DEPARTMENT OF MATHEMATICS

TECHNICAL REPORT

THE DENSE PACKING OF 13 CONGRUENT CIRCLES IN A CIRCLE

Ferenc Fodor

March 2001

No. 2001-1

TENNESSEE TECHNOLOGICAL UNIVERSITY
Cookeville, TN 38505

THE DENSEST PACKING OF 13 CONGRUENT CIRCLES IN A CIRCLE

FERENC FODOR

Abstract. The densest packings of n congruent circles in a circle are known for $n \leq 12$ and $n = 19$. In this paper we exhibit the densest packings of 13 congruent circles in a circle. We show that the optimal configurations are identical to Kravitz's [11] conjecture. We use a technique developed from a method of Bateman and Erdős [1] which proved fruitful in investigating the cases $n = 12$ and 19 by the author [6, 7] .

1. Preliminaries and Results

We shall denote the points of the Euclidean plane $\mathbf{E}^2$ by capitals, sets of points by script capitals, and the distance of two points by $d(P,Q)$. We use PQ for the line through P, Q, and $\overline{PQ}$ for the segment with endpoints P, Q. $\angle POQ$ denotes the angle determined by the three points P,O,Q in this order. $C(r)$ means the closed disc of radius r with center O. By an annulus $r < \rho \leq s$ we mean all points P such that $r < d(P,O) \leq s$. We utilize the linear structure of $\mathbf{E}^2$ by identifying each point P with the vector $\vec{OP}$, where O is the origin. For a point P and a vector $\vec{a}$ by $P + \vec{a}$ we always mean the vector $\vec{OP} + \vec{a}$.

The problem of finding the densest packing of congruent circles in a circle arose in the 1960s. The question was to find the smallest circle in which we can pack n congruent unit circles, or equivalently, the smallest circle in which we can place n points with mutual distances at least 1. Dense circle packings were first given by Kravitz [11] for $n = 2, \ldots, 16$. Pirl [14] proved that the these arrangements are optimal for $n \leq 9$ and he also found the optimal configuration for $n = 10$. Pirl also conjectured dense configurations for $11 \leq n \leq 19$. For $n \leq 6$ proofs were given independently by Graham [3]. A proof for $n = 6$ and 7 was also given by Crilly and Suen [4]. Subsequent improvements were presented by Goldberg [8] for $n = 14, 16$ and 17. He also found a new packing with 20 circles. In 1975 Reis [15] used a mechanical argument to generate remarkably good packings up to 25 circles. Recently, Graham et al. [9, 10] using computers established packings with more than 100 circles and improved the packing of 25 circles. In 1994 Melissen [12] proved Pirl's conjecture for $n = 11$ and the author [6, 7] proved it for $n = 19$ and $n = 12$. The problem of finding the densest packing of equal circles in a circle is also mentioned as an unsolved problem in the book of Croft, Falconer and Guy [5]. Packings of congruent circles in hyperbolic plane were treated by K. Bezdek [2]. Analogous results of packing n equal circles in an equilateral triangle and square can be traced down in the doctoral dissertation of Melissen [13].

1991 *Mathematics Subject Classification.* Primary52C15.

Key words and phrases. circle packings, density, optimal packing.

In this article we shall find the optimal configurations for $n = 13$. We are going to prove the following theorem.

Theorem 1. *The smallest circle $\mathcal{C}$ in which we can pack 13 points with mutual distances at least 1 has radius $R = (2\sin 36°)^{-1} = \frac{1+\sqrt{5}}{2}$. The 13 points form the following two configuration as shown on Figure 1.*

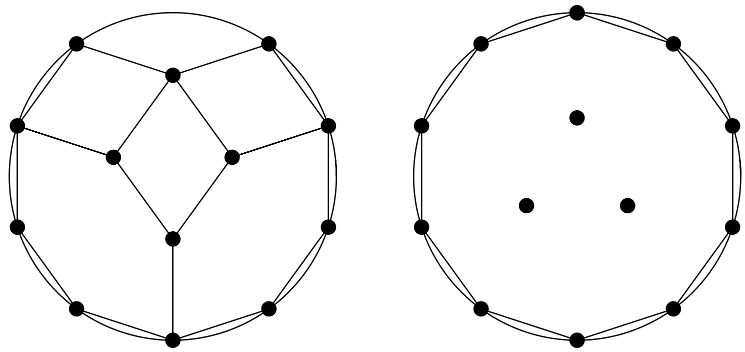

FIGURE 1. The optimal configurations for $n = 13$.

We shall prove Theorem 1 in the following way. We are going to show that it is possible to divide $C(R)$ into a smaller circle $C(S)$ and an annulus $S < \rho \leq R$, choosing S such that there can be at most 4 points in $C(S)$ and at most 10 points $S < \rho \leq R$. Then we prove that the 4 points in $C(S)$ form the configurations shown on Figure 1.

In the course of our proof we shall use the following two statements. Lemma 1, slightly modified here, originates from a paper of Bateman and Erdős [1]. Lemma 2 was used by the author in [6, 7].

Lemma 1 ([1])**.** *Let $r, s, (s \geq \frac{1}{2})$ be two positive real numbers and suppose that we have two points P and Q which lie in the annulus $r \leq \rho \leq s$ and which have mutual distance at least 1. Then the minimum $\phi(r,s)$ of the angle $\angle POQ$ has the following values:*

$$\phi(r,s) = \arccos \frac{s^2+r^2-1}{2rs}, \qquad \text{if } 0 < s-1 \leq r \leq s - 1/s;$$

$$\phi(r,s) = 2\arcsin \frac{1}{2s}, \qquad \text{if } 0 < s - 1/s \leq r \leq s \text{ or } s \leq 1.$$

Lemma 2 ([6])**.** *Let $\mathcal{S}$ be a set of n $(n \geq 2)$, points in the plane and $\mathcal{C}$ the smallest circle containing $\mathcal{S}$. Let $\vec{a}$ be a vector. There exist two points P_1, P_2 on the boundary of $\mathcal{S}$, such that $d(P_1 + \vec{a}, O) + d(P_2 + \vec{a}, O) \geq 2r$, where r is the radius and O is the center of $\mathcal{C}$.*

2. PROOFS

Let $S = R - 1/R = 1$. It was proved by Bateman and Erdős [1] that 7 points with mutual distances at least 1 can be packed into a unit circle in a unique way; one point is at the center of the circle, and the other 6 form a regular hexagon of unit side length with vertices on $C(1)$. There cannot be any further points situated in the annulus $1 < \rho \leq R$. The argument for 6 points is very similar. The radius of the circumcircle of 6 points with mutual distances at least 1 is 1, see [1]. Furthermore, the 6 points either form a regular hexagon of unit side length with vertices on $C(1)$, or a pentagon with all 5 vertices on $C(1)$ and a sixth point at O. In both cases it is clear that there cannot be 7 points in the annulus $1 < \rho \leq R$. Note that if there are exactly 9 points in the annulus $1 < \rho \leq R$, then the 13 points must form the first configuration shown on Figure 1.

Lemma 3. *There cannot be exactly 5 points in $C(1)$.*

Proof. Suppose, on the contrary, that there are 5 points in $C(1)$. Bateman and Erdős [1] proved that the radius of the circumcircle of 5 points with mutual distances at least 1 is $d_5 = (2\cos 54°)^{-1} = 0.85\ldots$. The minimal radius is realized by a regular pentagon of unit side length. According to Lemma 2 there must be two points P and Q of the 5 such that $d_P = d(P,O) \geq d_Q = d(Q,O)$ and $d_P + d_Q \geq 2d_5$.

Furthermore, the 4 points other than P cannot all be in $C(0.812)$. To see this suppose that they all are in $C(0.812)$. By adding up the five central angles we obtain $3\phi(0.812, 0.812) + 2\phi(1, 0.812) = 360°.12\ldots$. Therefore we may suppose that $d_P \leq 2d_5 - 0.812 = 0.8894\ldots$. Simple calculus shows that $2\phi(d_P, R) + 2\phi(d_Q, R)$ takes on its minimum, under these circumstances, when $d_P = 2d_5 - 0.812$ and $d_Q = 0.812$, and the minimum is $126.197°$.

Note further that not all 3 points other than P and Q can be in $C(0.73)$. To see this add up the 5 central angles, $\phi(1,1) + 2\phi(0.73, 0.73) + 2\phi(1, 0.73) = 370°\ldots$ if P and Q are adjacent. If they are not adjacent, then the sum is $\phi(0.73, 0.73) + 4\phi(1, 0.73) = 360°.8\ldots$. Hence, there is a point S such that $d_S = d(S,O) \geq 0.73$.

Let us also observe that $d_S \leq 0.762$ else the sum of the 8 central angles determined by the points in the annulus is $180° + 126°.2 + 2\phi(0.762, R) = 360°.035\ldots$. We can also conclude that it is not possible that the 2 remaining points are both in $C(0.71)$. This follows from the fact that the sum of the 5 central angles would be larger than $360°$. We must check two different scenarios. One is when P and Q are adjacent. Then the sum is either at least $\phi(1,1) + \phi(1, 0.762) + \phi(0.71, 0.71) + \phi(0.762, 0.71) + \phi(0.71, 1) = 60° + 67°.6 + 89°.5 + 85°.5 + 69°.2 = 371°\ldots$, or $\phi(1,1) + 2\phi(0.762, 0.71) + 2\phi(0.71, 1) = 60° + 2\cdot 85°.5 + 2\cdot 69°.2 = 369°\ldots$.

If P and Q are not adjacent, then the sum of the 5 central angles is either $2\phi(1, 0.762) + \phi(0.71, 0.71) + 2\phi(0.71, 1) = 2\cdot 67°.6 + 89°.5 + 2\cdot 69°.2 = 363°.1\ldots$, or $\phi(1, 0.762) + \phi(0.71, 0.762) + 3\phi(0.71, 1) = 67°.6 + 85°.5 + 3\cdot 69°.2 = 360°.7\ldots$.

Now, the sum of the 8 central angles is $4\cdot 36° + 2\phi(d_P, R) + 2\phi(d_Q, R) + 2\phi(0.73, R) + 2\phi(0.71, R) = 270 + 45° + 48°.8 = 363°\ldots$. Thus we have finished the proof of the lemma. $\square$

Proposition 1. *Let $f(r) = \phi(r,R) + \phi(r,s)$ and $\frac{\sqrt{2}}{2} \leq s \leq 1$ fixed. If $R - 1 \leq r \leq \min\{0.77, s\}$, then $f(r)$ is an increasing function of r.*

Proof. To see this evaluate the derivative of $f(r)$. Our goal is to show that

$$f'(r) = \frac{\frac{R-r^2}{r}}{\sqrt{(2rR)^2 - (r^2+R)^2}} - \frac{\frac{r^2-s^2+1}{r}}{\sqrt{(2rs)^2 - (s^2+r^2-1)^2}} > 0 \tag{1}$$

After rearrangement of the terms we obtain

$$\sqrt{\frac{(2rs)^2 - (s^2+r^2-1)^2}{(2rR)^2 - (r^2+R)^2}} > \frac{r^2-s^2+1}{R-r^2} \tag{2}$$

Notice that $\frac{r^2-s^2+1}{R-r^2}$ takes on its maximum if $r=s=0.77$, and the maximum is less than 1. On the left hand side we can see that we decrease $(2rs)^2-(s^2+r^2-1)^2$ if we replace s by r because it is an increasing function of s. Now, we are going to show that

$$\frac{(2r^2)^2-(2r^2-1)^2}{(2rR)^2-(r^2+R)^2}>1.$$

After simplification we obtain

$$(2r^2)^2-(2r^2-1)^2-((2rR)^2-(r^2+R)^2)=r^4-2Rr^2+R.$$

This polynomial is zero if $r=\sqrt{R-1}=0.78\ldots$, therefore for $r\in[R-1,0.77]$ it is positive. Thus, the inequality holds. □

Let the 4 points in $C(1)$ be labeled as $P_1,\ldots,P_4$ in clockwise direction. Let $d_i=d(P_i,O)$ and assume that d_1 is the largest of d_i, $i=1,\ldots,4$.

Proposition 2. *In each sector determined by two consecutive points in $C(1)$, there must be at least 2 points from the annulus $1<\rho\le R$.*

Proof. First, note that $d_i\le 0.77$, $i=2,3,4$ beacuse $4\phi(0.77,R)+7\cdot 36^\circ>361^\circ$. Suppose, on the contrary, that in P_iOP_{i+1} there is only one point. The sum of the 9 central angles is not less than $252^\circ+\phi(d_i,R)+\phi(d_i,d_{i+1})+\phi(d_{i+1},R)$.

If $i=1$, then we know that $d_2\le d_1$. Therefore by Proposition 1 the total angle is not less than $252^\circ+\phi(d_1,R)+\phi(d_1,R-1)+\phi(R-1,R)$. This function takes on its minimum at $d_=1$, where it is exactly 360°. If $i=4$, then $d_4\le d_1$ and so we may repeat the previous argument such that we obtain the same formula for the total angle.

If $i=2$ or 3, then assume that $d_i\ge d_{i+1}$. Then, by Proposition 1 the total angle is not less than $252^\circ+\phi(d_i,R)+\phi(d_i,R-1)+\phi(R-1,R)$. This function takes on its minimum at $d_i=R-1$ and $d_i=1$. It means that $d_i=R-1$ which gives 360° for the total angle, plus $2\phi(d_1,R)-36^\circ$ which makes the total larger than 360°. □

Now we will examine the positions of the points $P_1,\ldots,P_4$. We are going to prove that $d_1=1$. Let the 9 points in the annulus $1<\rho\le R$ be labeled by $P_5,\ldots,P_{13}$ in clockwise direction such that P_5 is adjacent to the segment P_1O. Let c_i denote the unit circle centered at P_i. Furthermore, let $Q_i=c_i\cap c_{i-1}$ and $Q_1=c_1\cap P_1O$, $Q_0=c_9\cap P_1O$ be points in $C(1)$. Let $\mathcal{R}$ be the region of $C(d_1)$ which is not covered by the circular discs C_i. This is where P_2,P_3,P_4 can be situated. The vertical line P_1O cuts $\mathcal{R}$ into two subregions, $\mathcal{R}=\mathcal{R}_1\cup\mathcal{R}_2$. We are going to examine the diameter of $\mathcal{R}_1$ and $\mathcal{R}_2$. We will show that if $d_1<1$, then $\mathcal{R}_i$, $i=1,2$ cannot accomodate two points. We are going to demonstrate this by proving that the diameter of $\mathcal{R}_i$, $i=1,2$ is less than 1. now, we try to maximize the diameter of $\mathcal{R}_1$. We assume that $P_5,\ldots,P_{13}$ are on the circle $C(R)$.

Lemma 4. *For a fixed value of $d_1\in[\frac{\sqrt{2}}{2},1]$, the arc of c_1 between Q_0 and Q_6 is the longest if $\angle P_1OP_6=36^\circ+\psi$, where $\psi=\phi(d_1,R)$.*

Proof. Notice that of two unit circles centered on $C(R)$, the one whose center makes the smaller central angle with $\overline{P_1O}$ provides the longer arc on c_1 if their intersection is in c_1. Let $\angle P_1OP_6=\alpha$. For a fixed d_1, $\alpha\in[36^\circ+\psi,108^\circ-\psi]$. If d_1 is fixed it is enough to show that the intersection of the two unit circles in the extreme positions is in c_1. Note that this intersection point $T(d_1)$ is always such that $\angle P_1OT(d_1)=72^\circ$. Let $S=c_1\cap OT(d_1)$ and let $s=d(S,O)$. It is

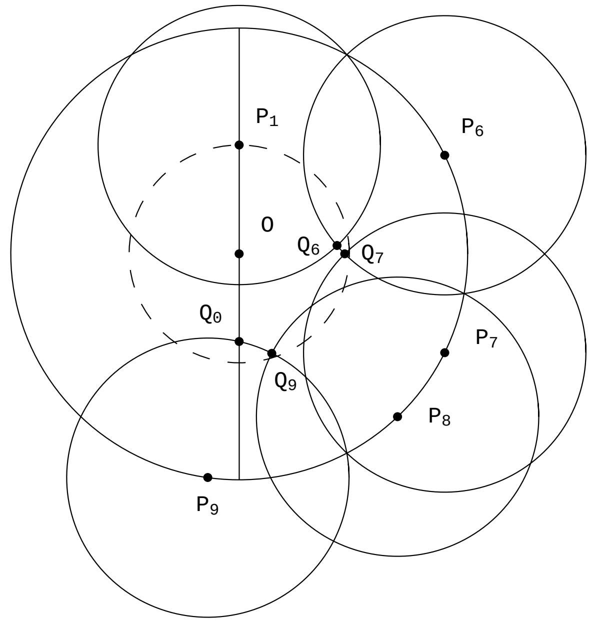

FIGURE 2

enough to show that $d^2(S, P_6) = R^2 + s^2 - 2Rs\cos(36° - \psi) \leq 1$. The equation $R^2 + s^2 - 2Rs\cos(36° - \psi) = 1$ can, after some transformations, be written as an algebraic equation for d_1 as follows.

$$(3\sqrt{5}+7)x^{12} + (-31-13\sqrt{5})x^{10} + (6\sqrt{5}+18)x^8 + (12\sqrt{5}+22)x^6 + (45+103\sqrt{5})x^4 + (-166-74\sqrt{5})x^2 + 47 + 21\sqrt{5} = 0$$

The above polyomial has only one root in $[\frac{\sqrt{2}}{2}, 1]$ and it is equal to 1. This proves our claim. $\square$

We assume that $\angle P_9OP_1 = 216° - \psi$, where $\psi = \phi(d_1, R)$. This ensures that the part of P_1O which bounds $\mathcal{R}_1$ is maximal in length. We will also assume that the angles are $\angle P_6OP_7 = \angle P_8OP_9 = 72° - \psi$. This guarantees that the arcs of c_7 and c_9 are the longest possible. However, notice that in such a position $d(P_7, P_8)$ becomes less than 1.

Lemma 5. *The diameter of* $\mathcal{R}_1$ *does not exceed* 1 *if* $d_1 \in [0.745, 1]$

Proof. Notice that $\angle P_1OQ_7 = 90°$ and $\angle P_1OQ_9 = 162°$, and $d(Q_7, O) = d(Q_9, O)$ is a decreasing function of d_1. Also, $d(Q_0, O)$ is a monotonically decreasing functions of d_1. $\mathcal{R}_1$ is bounded by arcs of $c_1, c_6, \ldots, c_9$, and the line P_1O. Its vertices are $Q_0, Q_1, Q_6, \ldots Q_9$ as shown on Figure 2.

Under these circumstances, the diameter of $\mathcal{R}_1$ can only be realized by a pair of the vertices Q_0, Q_6, Q_7, Q_9. Note that $d(Q_7, Q_9)$ and $d(Q_0, Q_7)$ are both decreasing functions of d_1, and by direct substitution we can see that they do not exceed 1 if $d_1 \in [0.745, 1]$.

Using the coordinates of Q_0 and Q_6 we may write $d^2(Q_0, Q_6) = 1$ as an algebraic equation for d_1. Furthermore, after a sufficient number of transformations,

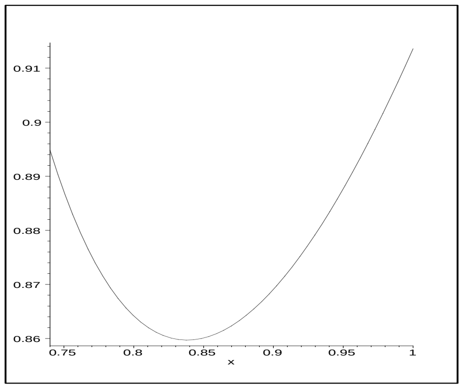

FIGURE 3

it may be written as a polynomial equation $p(Q_0, Q_6) = 0$ for d_1. The Cartesian coordinates of Q_0 and Q_6 are the following.

$Q_6 = (R(\sin(36° + \psi) - \sin\psi), d + R(\cos(36° + \psi) - \cos\psi));$

$Q_0 = (0, -R\cos(36° - \psi) + \sqrt{R^2\cos^2(36° - \psi) - R})$

The polynomial equation is as follows.

$p(Q_0, Q_6) = (-10 - 4\sqrt{5})d_1^4 + (-5 - 7\sqrt{5})d_1^6 + 14\sqrt{5}d_1^8 + (55 + 13\sqrt{5})d_1^{10} + (25 + 15\sqrt{5})d_1^{12} + (10 + 4\sqrt{5})d_1^{14} = 0$

This equation has two roots in the $[\frac{\sqrt{2}}{2}, 1]$ interval, $0.744\ldots$ and 1. By direct substitution we can check that $p(Q_0, Q_6) < 1$ in $(0.7448, 1)$. In a similar manner we may write $d^2(Q_6, Q_9) = 1$ as a polynomial equation for d_1 and check for roots in the designated interval. Note that $d(O, Q_9) = R\cos(36° - \psi) - \sqrt{(1 - R^2\sin^2(36° - \psi))}$. The graph of the function $d(Q_6, Q_9)$ is shown on Figure 2.

This function has no zeros in the interval $[0.745, 1]$. $\square$

Lemma 6. *If $P_9, P_{10}, P_{11} \in P_3OP_4$, then is not possible that $d_1 \in [\sqrt{2}/2, 0.745]$.*

Proof. For every value of d_1 there is a d_m such that none of the three points P_2, P_3, P_4 can be closer to O than d_m. We may obtain d_m from the the follwing equation.

$$\phi(d_1, d_1) + \phi(d_1, d_m) = 180°$$

Easy calculation shows that $d_m = \frac{1}{d_1} - d_1$. Clearly, d_m is a monotonically decreasing function of d_1. Furthermore, let $d_M = \sqrt{1 - d_1^2}$. We claim that it is not possible that both d_3 and d_4 are less than or equal to d_M. This may be shown simply by examining the function that describes the sum of the four central angles determined by the points $P_1, \ldots, P_4$. It is $f(d_1) = \phi(d_1, d_1) + 2\phi(d_M, d_1) + \phi(d_M, d_M)$. Simple calculus shows that

$$\frac{df(d_1)}{d\ d_1} = -2\frac{1}{d_1^2\sqrt{4-1/d_1^2}} + 2\frac{d_1}{(1-d_1^2)^{3/2}\sqrt{4-\frac{1}{1-d_1^2}}} \tag{3}$$

It is a simple exercise to show that (3) is larger than or equal to 0 in the interval $[\frac{\sqrt{2}}{2}, 1]$. In particular, $f(\frac{\sqrt{2}}{2}) = 360°$, which proves our claim. Also note that $\phi(d_M, d_1) = 90°$.

Now, we are going to add up the four central angles, $\angle P_iOP_{i+1}$. The sum of the angles is as follows $\phi(d_1, d_2) + \phi(d_2, d_3) + \phi(d_3, R) + 72° + \phi(d_4, R) + \phi(d_4, d_1)$. Note that $\phi(d_2, d_1)$ is not less than $\phi(d_1, d_1)$ and that by Lemma 2, $\phi(d_2, d_3) + \phi(d_3, R)$ takes on its minimum when $d_2 = d_1$ and $d_3 = d_m$. The same is true for $\phi(d_4, R) + \phi(d_4, d_1)$ so we may assume that $d_4 = d_M$. Whence, the sum of the angles is $\phi(d_1, d_1) + \phi(d_m, d_1) + \phi(d_m, R) + 72° + \phi(d_M, R) + \phi(d_M, d_1)$. After simplification we obtain

$$F(d_1) = \phi(d_M, R) + \phi(d_m, R) + 342°.$$

$F(d_1)$ is a decreasing function of d_1 and it is larger than 360° on the interval $[\sqrt{2}/2, 0.735]$.

In the interval $[0.735, 0745]$ we will write the total angle differently. First, notice that if $d_1 \geq \frac{1-R+\sqrt{6-R}}{2} = 0.7376\ldots$, then $d_m \leq R - 1$, so we may omit $\phi(d_m, R)$.
Case 1. $d_3 \geq 0.62$ The sum of the angles is not less than $\phi(d_1, d_1) + \phi(d_1, 0.62) + \phi(0.62, R) + \phi(d_M, R) + \phi(d_M, d_1) + 72°$, which is equal to $162° + \phi(0.62, R) + \phi(0.62, d_1) + \phi(d_1, d_1)$. This is a decreasig function of d_1 and its value at 0.745 is 360.2°.

Case 2. $d_3 < 0.62$ Notice that one of d_2 and d_4 has to be larger than or equal to 0.732 or $2\phi(0.62, 0.732) + 2\phi(0.732, 0.745) > 360°$. Moreover, none of d_2 and d_4 can be less than 0.72 or else $\phi(0.745, 0.745) + \phi(0.745, 0.62) + \phi(0.62, 0.72) + \phi(0.72, 0.745) > 360°$. In this case the total angle is not less than $2\phi(0.72, R) + 2\phi(0.732, R) + 2\phi(d_1, R) + 216° \geq 362°$.

□

Lemma 7. *If $P_{10}, P_{11}, P_{12} \in P_4OP_1$, then d_1 cannot be in $[\sqrt{2}/2, 0.745]$.*

Proof. The total angle is not less than $2\phi(d_1, d_1) + \phi(d_m, d_1) + \phi(d_m, R) + 72° + \phi(d_1, R)$ which is larger than or equal to $72° + 2\phi(d_1, d_1) + \phi(d_1, R) + \phi(R-1, d_1)$ in the $[0.737, 0.745]$ interval. This function is decreasing and its value exceeds $360°.2$ at $d_1 = 0.745$. Note that $\phi(d_1, d_1) + \phi(d_1, R)$ is a decreasing function in the designated interval.

In $[\sqrt{2}/2, 0.737]$, the total angle is larger than or equal to $252° + \phi(d_1, d_1) + \phi(d_m, R) + \phi(d_1, R)$ which is also a decreasing function of d_1 and its value at $d_1 = 0.737$ is 366° To see that this note that $\phi(d_m, R) + \phi(d_1, R)$ is a decreasing function of d_1. □

Now, the only possibility is that $d_1 = 1$. We saw in Lemma 5, that in this case the diameter of $\mathcal{R}_1$ and $\mathcal{R}_2$ is equal to 1 and this diameter is realized by Q_0 and

Q_6. Therefore the four points in $C(1)$ must be in the configuration shown in the second part of Figure 1.

References

[1] P. Bateman, P. Erdős, *Geometrical extrema suggested by a lemma of Besicovitch*, Amer. Math. Monthly, **58**, (1951), 306-314.

[2] K. Bezdek, *Ausfüllungen eines Kreises durch kongruente Kreise in der hyperbolischen Ebene*, Studia Sci. Math. Hungar, **17**, (1982), 353-356.

[3] H.S.M. Coxeter, M.G. Greening and R.L. Graham, *Sets of points with given maximum separation (Problem E1921)*, Amer. Math. Monthly, **75**, (1968), 192-193.

[4] T. Crilly, S. Suen, *An improbable game of darts*, Math. Gazette, **71**, (1987), 97-100.

[5] H.T. Croft, K.J. Falconer, R.K. Guy, *Unsolved Problems in geometry*, Springer Verlag New York, Berlin, Heidelberg, 1991.

[6] F. Fodor, *The densest packing of 19 congruent circles in a circle*, Geom. Dedicata **74**, (1999), 139–145.

[7] F. Fodor, *The densest packing of 12 congruent circles in a circle*, Beitrage Alg. Geom. **41**, (2000), 401-409.

[8] M. Goldberg, *Packing of* $14, 16, 17$ *and* 20 *circles in a circle*, Math Mag., **44**, (1971), 134-139.

[9] R.L. Graham, B.D. Lubachevsky, *Dense packings of* $3k(k+1)+1$ *equal disks in a circle for* $k = 1, 2, 3, 4$ *and* 5, Proc. First Int. Conf., "Computing and Combinatorics" COCOON'95, Springer Lecture Notes in Computer Science, **959** (1996), 303-312.

[10] R.L. Graham, B.D. Lubachevsky, *Curved hexagonal packings of equal disks in a circle*, Discrete Comp. Geom., **18**, (1997), 179-194.

[11] S. Kravitz, *Packing cylinders into cylindrical containers*