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CLIFFORD ALGEBRA SPACE SINGULARITIES OF INLINE PLANAR PLATFORMS

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Abstract

A Jacobian matrix of a general inline planar platform is studied. An inline planar platform is a manipulator with three legs, each with RPR joints, such that the revolute joints are free and align on each platform and the prismatic joints are powered. The configurations that cause the Jacobian matrix to become singular form a singularity surface that must be avoided for controllability. The Jacobian matrix is developed in the even Clifford algebra $C\ell^+(P^2)$ of the projective space P^2 and its singularity surface is studied. A redundant planar platform manipulator is shown to have a block Jacobian matrix. A composite of singularity sets is developed for a redundant planar platform. A three-dimensional multi-platform manipulator is discussed.

Keywords: Clifford algebras, dual quaternions, hyper-redundant robots, Jacobian matrix, planar platform, singularity surface.

1 Introduction

Hyper-redundant robots are manipulators that have large or infinite number of degrees of kinematic redundancy [1]. For spatial robots, this means they have more than six degrees of freedom (DOF's) required for the end effector to reach, with any orientation, a given point in a dextrous workspace. A hyper-redundant robot can take on large or infinite number of shapes for a particular orientation and position of the end effector. For this reason hyper-redundant robots are ideal for working around obstacles or in confined workspaces. These manipulators may not have a well-defined end effector, since for some applications the links of the manipulator are used as a gripper, which can handle objects delicately, much the same way elephants can pick objects up with their trunks [8]. Two common methods of constructing these robots are a *serial link* and a *variable geometry truss structure* (VGT). Very little work has been done to solve the singularity configurations for VGT manipulators due to their complicated structure [4]. Jacobian matrices which determine the singularities of the end effector's velocity and force do not reveal the singularities of the joint configurations, which are more important since there are more of them and the shape of the hyper-redundant robot is as important as knowing the end effector's position.

Singularity sets for four revolute joint manipulators were studied by Long, McCarthy, and Paul [5] using screw theory, but only the position and orientation of the end effector were considered. Ge

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and McCarthy [3] found a constraint manifold for assembly line tolerance using a certain Clifford algebra [6]. Later, Collins and McCarthy [2] used the same Clifford algebra to develop singularity sets for planar platforms.

The planar VGT hyper-redundant robot can be constructed by stacking planar platforms. Planar platforms are a better choice than serial link manipulators because of better structural rigidity, kinematic accuracy, and dynamic control [10]. Better structural rigidity is needed because a hyper-redundant robot is heavy due to a large number of actuators. Better kinematic and dynamic accuracy is needed because the increase in DOF's leads to a decrease in positioning accuracy.

1.1 VGT Singularities

Collins and McCarthy [2] found a Clifford algebra representation of the ρ_{1i}'s, the lengths of the legs of a single platform, and their time derivatives along with a time derivative of a constraint, to form a square Jacobian matrix. The determinant of the Jacobian, when set to zero, parameterizes the singularity set of the platform. The work of Collins and McCarthy can be extended to find all singularities of the hyper-redundant robots described above. The objective of this research is to find and study singularity sets of two stacked planar platform manipulators shown in Fig. 1. A Jacobian that parameterizes the two stacked platforms must also parameterize the individual

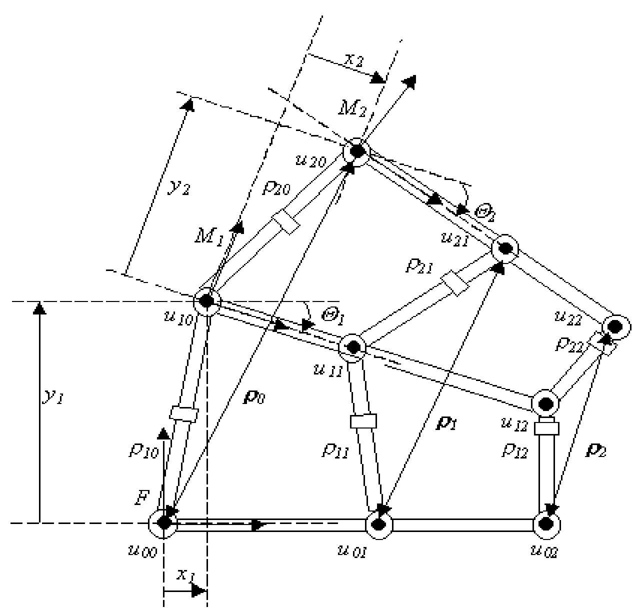

Figure 1: Two Stacked General Planar Platforms

platform singularities along with the singularities of the end effector. Only then it will be useful in the study of the hyper-redundant robots.

1.2 Paper Organization

The background information in Section 2 covers the singularity types for parallel manipulators. Then, a particular Clifford algebra needed to represent spatial rotations and translations is constructed. Section 3 develops the singularity set for one platform manipulator as a review of the work of Collins and McCarthy [2]. Section 4 develops the case for two stacked platforms. Conclusions and recommendations can be found in Section 5.

2 Singularity Types and Clifford Algebras

Singular configurations should be avoided in robot arms. At a singularity a robot can develop sufficiently large forces and torques that can cause damage to itself or to the environment; small perturbations in the link parameters can cause points in the workspace to be unreachable; there may be no solution or an infinite number of solutions to the inverse kinematics problem; the number of degrees of freedom (DOF's) of the robot will then change which will lead to a loss of controllability, and the end effector motion may become unattainable or require infinite joint velocities [10, 11].

2.1 Type I, Type II, and Type III Singularities

Gosselin and Angeles [4] have identified and categorized singularities of parallel manipulators into three types. Let $\boldsymbol{\theta}$ be a vector representing joint coordinates, either an angle for a revolute joint or a length for a prismatic joint. Let $\mathbf{x}$ be a vector in Cartesian coordinates representing a position (or orientation) of the manipulator [10]. A Jacobian is formed by taking the time derivative of a relation of the forward kinematics, given as the vector equation $F(\boldsymbol{\theta}, \mathbf{x}) = 0$, to form the equation $A\dot{\mathbf{x}} + B\dot{\boldsymbol{\theta}} = 0$, where A and B are matrices of partial derivatives with respect to $\mathbf{x}$ and $\boldsymbol{\theta}$. Singularities occur when either matrix A or B becomes *singular*, i.e., when the rank is no longer maximum. Singularity of type I occurs when B becomes singular, singularity of type II occurs when A becomes singular, and singularity of type III occurs when both matrices A and B are simultaneously singular.

Type I singularities occur on the boundary of the workspace. The end effector loses one or more DOF's in such a way that one or more force(s) or torque(s) can be applied to the end effector without needing to apply force or torque at the powered joint(s). Type III singularities require that certain linkages have the same length so they can be aligned. This condition can be easily avoided in the design phase. Type II singularities are more elusive: In this configuration, the end effector position can be locally movable with all the input joints locked. When the manipulator has many links of different lengths, these singularities become impossible to determine by inspection. Since type I and III singularities can easily be avoided, type II singularities are the most interesting for us to study.

Type II singularities of parallel manipulators were parameterized using the Jacobian matrix of the system by Sefrioui and Gosselin [10]. Singularity surfaces in the projective plane of the even Clifford algebra for three revolute-prismatic-revolute (RPR) legged planar platforms were studied by Collins and McCarthy [2].

2.2 The Clifford Algebra $C\ell^+(P^3)$

Ge and McCarthy [3] used the Clifford algebra $C\ell(P^3)$ of a three-dimensional projective space P^3 with the quadratic form

$$Q = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{1}$$

A general element $\mathbf{H}$ in $C\ell^+(P^3)$ is then given as

$$\mathbf{H} = h_1\mathbf{e}_2\mathbf{e}_3 + h_2\mathbf{e}_3\mathbf{e}_1 + h_3\mathbf{e}_1\mathbf{e}_2 + h_4 + h_1^0\mathbf{e}_4\mathbf{e}_1 + h_2^0\mathbf{e}_4\mathbf{e}_2 + h_3^0\mathbf{e}_4\mathbf{e}_3 + h_4^0\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\mathbf{e}_4, \tag{2}$$

where the h_i's and the h_i^0's are real and $\mathbf{e}_4^2 = 0$. $\mathbf{H}$ can also be written as a *dual quaternion*, namely,

$$\mathbf{H} = h_1 i + h_2 j + h_3 k + h_4 + h_1^0\varepsilon i + h_2^0\varepsilon j + h_3^0\varepsilon k + h_4^0\varepsilon, \tag{3}$$

where

$$\varepsilon = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\mathbf{e}_4, \quad i = \mathbf{e}_2\mathbf{e}_3, \quad j = \mathbf{e}_3\mathbf{e}_1, \quad k = \mathbf{e}_1\mathbf{e}_2, \tag{4a}$$

$$\varepsilon i = \mathbf{e}_4\mathbf{e}_1, \quad \varepsilon j = \mathbf{e}_4\mathbf{e}_2, \quad \varepsilon k = \mathbf{e}_4\mathbf{e}_3. \tag{4b}$$

Quaternions are generalized complex numbers where the imaginary elements multiply as $i^2 = j^2 = k^2 = -1$, $ij = k = -ji$, $jk = i = -kj$, and $ki = j = -ik$ [6]. The dual element ε squares to zero and commutes with i, j, and k. As defined, the dual quaternion (3) is an element of $C\ell^+(P^3)$.

The case for planar motions requires that the third basis element $\mathbf{e}_3$ be nonexistent. Without the terms with $\mathbf{e}_3$, (2) reduces to a projection onto the even Clifford subalgebra $C\ell^+(P^2)$

$$\mathbf{H} = h_1^0 \underbrace{\mathbf{e}_4\mathbf{e}_1}_{\varepsilon i} + h_2^0 \underbrace{\mathbf{e}_4\mathbf{e}_2}_{\varepsilon j} + h_3 \underbrace{\mathbf{e}_1\mathbf{e}_2}_{k} + h_4, \tag{5}$$

and then (3) gives a *planar quaternion*

$$\mathbf{H} = h_1^0\varepsilon i + h_2^0\varepsilon j + h_3 k + h_4. \tag{6}$$

2.3 $C\ell^+(P^3)$ Components from the Screw Parameterization

Chasles Theorem. *Every rigid body motion can be realized by a rotation about an axis combined with a translation parallel to that axis.*

Any rigid body displacement can be represented by screws. A *screw* consists of a rotation by an angle θ_1 about an axis $\mathbf{L}$, and a translation along the same axis by a distance d (see [9]). Let F be a fixed frame and let M be a moving frame attached to the end effector. The homogeneous transformation matrix which describes M in F coordinates is given by

$$T = \begin{bmatrix} A & \mathbf{d} \\ 0 & 1 \end{bmatrix}, \tag{7}$$

where A is a 3×3 orthogonal matrix that describes the orientation of M relative to F, and $\mathbf{d}$ is a 3×1 translation vector that locates the position of the origin of M relative to F. To describe the screw motion of M, a unit vector $\mathbf{s} = (s_x, s_y, s_z)$ on $\mathbf{L}$ and the rotation angle θ_1 about $\mathbf{L}$ can be found from the matrix A as

$$\cos(\theta_1) = \frac{1}{2}(a_{11} + a_{22} + a_{33} - 1), \tag{8}$$

$$s_x = \frac{a_{23} - a_{32}}{2\sin(\theta_1)}, \quad s_y = \frac{a_{31} - a_{13}}{2\sin(\theta_1)}, \quad s_z = \frac{a_{12} - a_{21}}{2\sin(\theta_1)}, \tag{9}$$

where a_{ij} is the ij-th element of A. Since A is orthogonal, McCarthy [7] used Cayley's formula to find Euler parameters of the rotation as

$$\mathbf{h} = \begin{bmatrix} s_x \sin(\frac{\theta_1}{2}) \\ s_y \sin(\frac{\theta_1}{2}) \\ s_z \sin(\frac{\theta_1}{2}) \\ \cos(\frac{\theta_1}{2}) \end{bmatrix}, \tag{10}$$

(here $\mathbf{h} = (h_1, h_2, h_3, h_4)$ from (2)). Since $\mathbf{s} = [s_x, s_y, s_z]$ is a unit vector, vector $\mathbf{h}$ satisfies the constraint $\mathbf{h} \cdot \mathbf{h} = 1$ where $\cdot$ is the usual Euclidean dot product. Let $\mathbf{c}$ be a position vector from the origin of F to some point on $\mathbf{L}$ and let $\mathbf{s}^* = \mathbf{c} \times \mathbf{s}$. Ge and McCarthy [3] found the components of $\mathbf{h}^0 = (h_1^0, h_2^0, h_3^0, h_4^0)$ to be

$$\mathbf{h}^0 = \begin{bmatrix} \frac{\theta_1}{2} s_x \cos(\frac{\theta_1}{2}) + s_x^* \sin(\frac{\theta_1}{2}) \\ \frac{\theta_1}{2} s_y \cos(\frac{\theta_1}{2}) + s_y^* \sin(\frac{\theta_1}{2}) \\ \frac{\theta_1}{2} s_z \cos(\frac{\theta_1}{2}) + s_z^* \sin(\frac{\theta_1}{2}) \\ -\frac{\theta_1}{2} \sin(\frac{\theta_1}{2}) \end{bmatrix}. \tag{11}$$

The outer product $\mathbf{h} \times \mathbf{h}^0 = 0$ is a second constraint on $\mathbf{H}$. Then the *dual quaternion* $\mathbf{H} = (\mathbf{h}, \mathbf{h}^0)$ has six free parameters because of the two constraints.

In the case of planar displacements, where the z axes of M and F are perpendicular to the plane of motion, (10) and (11) reduce to

$$\mathbf{h} = \begin{bmatrix} 0 \\ 0 \\ \sin(\frac{\theta_1}{2}) \\ \cos(\frac{\theta_1}{2}) \end{bmatrix}, \quad \text{and} \quad \mathbf{h}^0 = \begin{bmatrix} \frac{1}{2} x_1 \cos(\frac{\theta_1}{2}) + \frac{1}{2} y_1 \sin(\frac{\theta_1}{2}) \\ -\frac{1}{2} x_1 \sin(\frac{\theta_1}{2}) + \frac{1}{2} y_1 \cos(\frac{\theta_1}{2}) \\ 0 \\ 0 \end{bmatrix}. \tag{12}$$

Since the first two components of $\mathbf{h}$ and the second two components of $\mathbf{h}^0$ are always zero for planar displacements, Ge and McCarthy [7] use

$$\mathbf{q} = \begin{bmatrix} \frac{1}{2} x_1 \cos(\frac{\theta_1}{2}) + \frac{1}{2} y_1 \sin(\frac{\theta_1}{2}) \\ -\frac{1}{2} x_1 \sin(\frac{\theta_1}{2}) + \frac{1}{2} y_1 \cos(\frac{\theta_1}{2}) \\ \sin(\frac{\theta_1}{2}) \\ \cos(\frac{\theta_1}{2}) \end{bmatrix} \tag{13}$$

to represent the position of M relative to F. Putting $\mathbf{q} = (q_1, q_2, q_3, q_4)$ into (6) gives a dual planar quaternion

$$\mathbf{q} = q_1 \varepsilon i + q_2 \varepsilon j + q_3 k + q_4. \tag{14}$$

3 Singularity Surfaces in $C\ell^+(P^2)$ of Single Stage Planar Platforms

3.1 The Homogeneous Transformations

In the planar case, matrix T of the homogeneous transformation in (7) becomes the 3×3 matrix in

$$\begin{bmatrix} X \\ Y \\ 1 \end{bmatrix} = \begin{bmatrix} \cos(\theta_1) & -\sin(\theta_1) & x_1 \\ \sin(\theta_1) & \cos(\theta_1) & y_1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \tag{15}$$

where θ_1 is the rotation of the moving frame about the z-axis of the fixed frame and (x_1, y_1) is the translation vector of the origin of the moving frame with respect to the origin of the fixed frame. The same homogeneous transformation matrix can be written using the quaternionic components of $\mathbf{q}$ as[1]

$$\begin{bmatrix} X \\ Y \\ 1 \end{bmatrix} = \begin{bmatrix} q_4^2 - q_3^2 & -2q_3q_4 & 2(q_1q_4 - q_2q_3) \\ 2q_3q_4 & q_4^2 - q_3^2 & 2(q_1q_3 + q_2q_4) \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}. \tag{16}$$

3.2 The Constraint Manifold

A general RPR planar platform is shown in Fig. 2, where the x_{ij}'s and y_{ij}'s are treated as constants. For any point (x_1, y_1, θ_1) in the workspace that is not a singularity, unique inputs ρ_0, ρ_1, ρ_2 can be found which means that this manipulator can be controlled by powering up the three prismatic joints. The triangles EFG and PQR connect the pivots in the F and M frames and represent rigid bodies. The $\mathbf{u}_{0i}$ position vectors given in the F coordinate frame and the $\mathbf{u}_{1i}$ position vectors given in the M coordinate frame are

$$\begin{aligned} \mathbf{u}_{00} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \mathbf{u}_{01} = \begin{bmatrix} x_{01} \\ 0 \end{bmatrix}, \quad \mathbf{u}_{02} = \begin{bmatrix} x_{02} \\ y_{02} \end{bmatrix}, \\ \mathbf{u}_{10} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \mathbf{u}_{11} = \begin{bmatrix} x_{11} \\ 0 \end{bmatrix}, \quad \mathbf{u}_{12} = \begin{bmatrix} x_{12} \\ y_{12} \end{bmatrix}. \end{aligned} \tag{17}$$

Collins and McCarthy [2] defined the constraint manifold by first defining three distances, ρ_i, $i = 0, 1, 2$, representing the lengths of the RPR chains as

$$\langle (\mathbf{u}_{0i} - \mathbf{U}_{1i}), (\mathbf{u}_{0i} - \mathbf{U}_{1i}) \rangle = (\mathbf{u}_{0i} - \mathbf{U}_{1i}) \cdot (\mathbf{u}_{0i} - \mathbf{U}_{1i}) = \rho_i^2, \tag{18}$$

[1] The transformation matrix in (16) is a corrected version of the transformation matrix displayed in (10) from [2] where the factor of 2 was omitted in the entries (1,3) and (2,3).

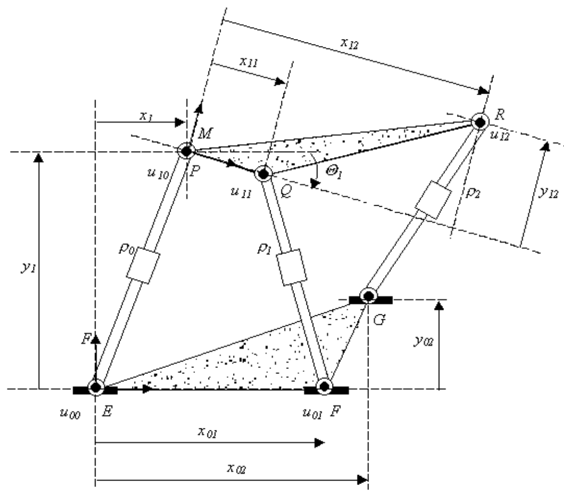

Figure 2: General Planar Platform

where $\mathbf{U}_{1i}$ is the position of the u_{1i} joint in the F coordinate frame found using the transformation matrix of (16). Therefore, for each $i = 0, 1, 2$,

$$\begin{bmatrix} x_{0i} \\ y_{0i} \\ 1 \end{bmatrix} - \begin{bmatrix} q_4^2 - q_3^2 & -2q_3q_4 & 2(q_1q_4 - q_2q_3) \\ 2q_3q_4 & q_4^2 - q_3^2 & 2(q_1q_3 + q_2q_4) \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_{1i} \\ y_{1i} \\ 1 \end{bmatrix} = \mathbf{u}_{0i} - \mathbf{U}_{1i}, \tag{19}$$

and (18) gives the following quadratic equation in the components of $\mathbf{q}$:

$$a_iq_1^2 + b_iq_2^2 + c_iq_3^2 + d_iq_4^2 + 2f_iq_2q_3 + 2g_iq_1q_3 + 2h_iq_1q_2 + 2l_iq_1q_4 + 2m_iq_2q_4 + 2n_iq_3q_4 = \frac{1}{4}\rho_i^2, \tag{20}$$

where[2]

$$\begin{bmatrix} a_i \\ b_i \\ c_i \\ d_i \\ f_i \\ g_i \\ h_i \\ l_i \\ m_i \\ n_i \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ \frac{1}{4}(x_{1i} + x_{0i})^2 + \frac{1}{4}(y_{1i} + y_{0i})^2 \\ \frac{1}{4}(x_{1i} - x_{0i})^2 + \frac{1}{4}(y_{1i} - y_{0i})^2 \\ \frac{1}{2}(x_{1i} + x_{0i}) \\ -\frac{1}{2}(y_{1i} + y_{0i}) \\ 0 \\ \frac{1}{2}(x_{1i} - x_{0i}) \\ \frac{1}{2}(y_{1i} - y_{0i}) \\ \frac{1}{2}(y_{1i}x_{0i} - x_{1i}y_{0i}) \end{bmatrix}. \tag{21}$$

[2]There is a sign error in [2], equation (15), in the expression for g_i.

3.3 The Quaternionic Jacobian

A Jacobian was found in [2] by taking the time derivative of each of the three equations given by (20) and the time derivative of the rigid body motion constraint

$$q_3^2 + q_4^2 = 1. \tag{22}$$

The derivative of (20) can be expressed as[3]

$$2\mathbf{q}^{\mathrm{T}} C_i \dot{\mathbf{q}} = \frac{1}{2}\rho_i \dot{\rho}_i, \quad i = 0, 1, 2, \tag{23}$$

where C_i is given by

$$C_i = \begin{bmatrix} a_i & h_i & g_i & l_i \\ h_i & b_i & f_i & m_i \\ g_i & f_i & c_i & n_i \\ l_i & m_i & n_i & d_i \end{bmatrix}. \tag{24}$$

The time derivative of (22) can be expressed as

$$[0 \quad 0 \quad q_3 \quad q_4]\dot{\mathbf{q}} = 0. \tag{25}$$

Combining three equations (23) with (25), we get the following Jacobian:

$$A\dot{\mathbf{q}} - B\dot{\mathbf{r}} = 0, \tag{26}$$

where

$$A = \begin{bmatrix} & \mathbf{q}^T C_0 & \\ & \mathbf{q}^T C_1 & \\ & \mathbf{q}^T C_2 & \\ 0 \quad 0 & q_3 & q_4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} \frac{1}{4}\rho_0 & 0 & 0 & 0 \\ 0 & \frac{1}{4}\rho_1 & 0 & 0 \\ 0 & 0 & \frac{1}{4}\rho_2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{27}$$

3.4 Singularity Sets of Planar Platforms

When matrix A is singular, a type II singularity occurs. Since A in (27) is square, it becomes singular whenever

$$\det A = 0. \tag{28}$$

3.4.1 General Planar Platform

A general planar platform is shown in Fig. 2. Setting the determinant of A from (27) to zero yields

$$S: A_1 q_1^2 q_3^2 + A_2 q_1^2 q_3 q_4 + A_3 q_2^2 q_4^2 + A_4 q_2^2 q_3 q_4 + A_5 q_1 q_3^3 + A_6 q_2 q_4^3 + A_7 q_1 q_2 q_3^2 + A_8 q_1 q_2 q_4^2 + A_9 q_1 q_3^2 q_4 + A_{10} q_1 q_3 q_4^2 + A_{11} q_2 q_3^2 q_4 + A_{12} q_2 q_3 q_4^2 = 0, \tag{29}$$

[3] The T in (23) denotes the matrix transpose.

where A_1 through A_{12} are given by

$$\begin{bmatrix} x_{11}x_{02} - x_{01}x_{12} \\ -(x_{11}y_{02} + x_{01}y_{12}) \\ -(x_{11}x_{02} - x_{01}x_{12}) \\ -(x_{11}y_{02} + x_{01}y_{12}) \\ -\frac{1}{2}(y_{12}x_{02} - x_{12}y_{02})(x_{01} + x_{11}) \\ \frac{1}{2}(y_{12}x_{02} - x_{12}y_{02})(x_{01} - x_{11}) \\ x_{11}y_{02} - x_{01}y_{12} \\ x_{11}y_{02} - x_{01}y_{12} \\ x_{01}x_{02}(x_{12} - x_{11}) + x_{11}x_{12}(x_{02} - x_{01}) + x_{01}y_{02}y_{12} + x_{11}y_{12}y_{02} \\ -x_{01}y_{02}(\frac{1}{2}x_{12} - x_{11}) + x_{11}y_{12}(\frac{1}{2}x_{02} - x_{01}) + \frac{1}{2}x_{01}x_{02}y_{12} - \frac{1}{2}x_{11}x_{12}y_{02} \\ x_{01}y_{02}(\frac{1}{2}x_{12} - x_{11}) + x_{11}y_{12}(\frac{1}{2}x_{02} - x_{01}) - \frac{1}{2}x_{01}x_{02}y_{12} - \frac{1}{2}x_{11}x_{12}y_{02} \\ x_{01}x_{02}(x_{12} - x_{11}) - x_{11}x_{12}(x_{02} - x_{01}) + x_{01}y_{02}y_{12} - x_{11}y_{12}y_{02} \end{bmatrix}.$$

Collins and McCarthy [2] call S a *singularity surface* in q_1, q_2, q_3, and q_4 coordinates. In order to visualize this surface in three dimensions, (29) can be divided by q_4^4, since $q_4 > 0$ assuming that θ_1 is limited to $-\pi < \theta_1 < \pi$. Substituting

$$x = \frac{q_1}{q_4}, \quad y = \frac{q_2}{q_4}, \quad z = \frac{q_3}{q_4} \tag{30}$$

into (29) yields

$$A_1x^2z^2 + A_2x^2z + A_3y^2 + A_4y^2z + A_5xz^3 + A_6y + A_7xyz^2 \\ + A_8xy + A_9xz^2 + A_{10}xz + A_{11}yz^2 + A_{12}yz = 0. \tag{31}$$

For specific values of z, (31) gives a quadric curve in the x, y plane [2].

3.4.2 In-Line Planar Platforms

An *in-line planar platform* has the three pivots aligned for both the base and the top. Therefore the dimensions y_{02} and y_{12} are zero. Then the A matrix of (27) becomes a block matrix[4]

$$A = \begin{bmatrix} A_1 & | & A_2 \end{bmatrix}. \tag{32}$$

where

$$A_1 = \begin{bmatrix} q_1 & q_2 \\ q_1 + \frac{1}{2}(x_{11} - x_{01})q_4 & q_2 + \frac{1}{2}(x_{11} + x_{01})q_3 \\ q_1 + \frac{1}{2}(x_{12} - x_{02})q_4 & q_2 + \frac{1}{2}(x_{12} + x_{02})q_3 \\ 0 & 0 \end{bmatrix}$$

[4]Equation (32) is a corrected version of Equation (36) in [2] where q_2 should have been q_4 in the (2,1) and (3,1) matrix elements.

and

$$A_2 = \begin{bmatrix} 0 & 0 \\ \frac{1}{2}(x_{11}+x_{01})q_2 + \frac{1}{4}(x_{11}+x_{01})^2 q_3 & \frac{1}{2}(x_{11}-x_{01})q_1 + \frac{1}{4}(x_{11}-x_{01})^2 q_4 \\ \frac{1}{2}(x_{12}+x_{02})q_2 + \frac{1}{4}(x_{12}+x_{02})^2 q_3 & \frac{1}{2}(x_{12}-x_{02})q_1 + \frac{1}{4}(x_{12}-x_{02})^2 q_4 \\ q_3 & q_4 \end{bmatrix}.$$

The singularity set is then given by (28) as

$$S : A_P(q_1^2 q_3^2 - q_2^2 q_4^2) + B_T(q_2 q_3 q_4^2 + q_1 q_3^2 q_4) + C_T(q_2 q_3 q_4^2 - q_1 q_3^2 q_4) = 0, \quad (33)$$

where

$$A_P = x_{02}x_{11} - x_{01}x_{12}, \quad B_T = x_{01}x_{02}(x_{12} - x_{11}), \quad C_T = -x_{11}x_{12}(x_{02} - x_{01}). \quad (34)$$

3.4.3 Singularity set of a general in-line planar platform

A general in-line platform with $x_{11} = 1, x_{12} = 4, x_{01} = 3$, and $x_{02} = 5$ was studied in [2] and [10]. Substituting these values and (30) in (33) gives equation

$$-7x^2z^2 + 53xz^2 + 7y^2 + 37yz = 0, \quad (35)$$

determines two surfaces. To find equations of these surfaces, factor (35) by solving it as a quadratic equation in y, which yields

$$y = -\frac{z}{14}(37 \pm \sqrt{196x^2 - 1484x + 1369}), \quad (36)$$

and the surfaces are given by

$$(y + \frac{z}{14}(37 - \sqrt{\beta}))(y + \frac{z}{14}(37 + \sqrt{\beta})) = 0, \quad (37)$$

where $\beta = 196x^2 - 1484x + 1369$. The discriminant β is quadratic in x, and setting it equal to 0 yields two values for $x : x_a = 1.0752$ and $x_b = 6.4962$. It can be easily seen that the discriminant β is negative when $x_a < x < x_b$, and it is positive when $x < x_a$ or $x > x_b$. From (37) or (35) it follows that if z were zero then it would not matter if the discriminant β were negative. When z is zero, $y = 0$ is the only solution to (37), so the line $y = 0, z = 0$, that is the x axis, gives additional singular configurations which should be included, even when the discriminant of (37) is less than zero. Fig. 3 shows the entire type II singular configuration for the general in-line planar platform.

4 Singularity Surfaces in $C\ell^+(P^2)$ of Two Stacked Planar Platform

The singular configurations in quaternion form for planar platforms were identified in Section 3. The singular configurations for two stacked planar platforms must also include the singular configurations of the separate platforms since, at these configurations, it is impossible to know the position of the

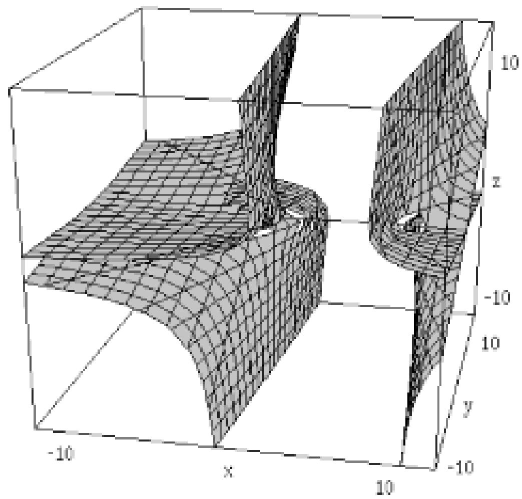

Figure 3: General In-Line Planar Platform Singularity Surface

moving pivots. If these are the only singular configurations, then the singularity set for stacked planar platforms is the union of the individual singularity sets. Two stacked planar platforms can be viewed as two links of a serial link manipulator, which could have singular positions due to the position of one link relative to the other. If there are additional singularities, then there should exist a Jacobian matrix that identifies these singular configurations when it becomes singular.

Two stacked planar platforms could be constructed as in Fig. 1, where ρ_0, ρ_1, and ρ_2 are treated as the inputs of the manipulator. A hyper-redundant manipulator can then be built by stacking many two-stacked manipulators together since the pivots at the top of the second platform align with the pivots on the bottom of the first.

4.1 The Jacobian of Two Stacked General In-Line Planar Platforms

Considering ρ_0, ρ_1, and ρ_2 in Fig. 1 as inputs, a 12×12 Jacobian matrix could be constructed as before. Observe that if a planar platform is not at or near a singular configuration, then the output is unique and any planar motion is possible. If both platforms in Fig. 1 are able to move in any planar direction and their configurations are stable, then clearly the outputs θ_1, x_1, and y_1 are stable and the end effector can move in any direction in the plane. So the manipulator is not in a singular configuration. Thus it has been shown that the distances ρ_0, ρ_1, and ρ_2 (see Fig. 1) should not be considered as inputs and the only inputs are the six ρ_{ij}s (three from each platform). Furthermore, there are no additional singular configurations for the two stacked platforms other than the union of the individual singular configurations. The assumption of viewing the two stacked platform as analogous to a two-link serial robot arm which would add singular configurations is not appropriate, since they are type I for the two-link planar case. The singular configurations studied here are of type II (see Section 2).

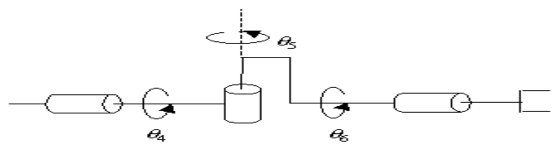

Figure 4: Spherical Wrist

Three platforms can be joined at the sides to form a three-dimensional platform similar to a Stewart platform. It may be possible, by stacking these platforms, to involve a type II singularity when the individual platforms are free of singularities. A comparison can be made to the spherical wrist, shown in Fig. 4. It is made up of three powered revolute joints. When θ_5 is such that the axis of the other two joints align, then there is an infinite number of solutions for θ_4 and θ_6 in the inverse kinematics problem. Therefore this configuration is in a singularity. A similar situation occurs when three of the three-dimensional platforms are stacked.

5 Conclusions and Recommendations

A study of the type II singular configurations of stacked planar platforms was completed. The singularity surface of the general in-line planar platform was corrected and significantly reduced, which is important since it is an important type of planar platform to be linked together to form a hyper-redundant manipulator and thus allows more freedom of movement. The singular configurations of a stacked planar platform were found to be the union of those of the individual platforms.

There are several recommendations for further work. The three-dimensional platform, such as a Stewart platform, possibly could be analyzed using dual quaternions. Also, the workspace of a manipulator is as important as the singularity position surface. Finding the workspace in quaternion form and subtracting the singular configurations from it would identify separate regions such that the manipulator could not go from one to another without external help. Another area for further work could be in finding a way to do all the computations algebraically in a program that could do Clifford algebra, without the use of matrices.

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