DEPARTMENT OF MATHEMATICS

TECHNICAL REPORT

TWO RESEARCH TRADITIONS SEPARATED BY A COMMON SUBJECT: MATHEMATICS AND MATHEMATICS EDUCATION

DR. ANNIE SELDEN

AUGUST 2002

No. 2002-2

TENNESSEE TECHNOLOGICAL UNIVERSITY
Cookeville, TN 38505

Two Research Traditions Separated by a Common Subject: Mathematics and Mathematics Education[1]

Annie Selden
Mathematics Department
Tennessee Technological University

Abstract

"*There are no proofs in mathematics education.*"[2] *While this is true, claims are made in mathematics education research and evidence is provided for them. In this talk, I will explore the nature of such research, the kinds of claims and evidence, and what such research might have to offer teachers of mathematics, especially at the undergraduate level. Along the way, I will point out differences between the ways research is done in the two fields.*

The above title is meant to be both provocative and descriptive. While I will talk about how research in the two fields -- mathematics and mathematics education -- differs, I would like to note, at the outset, something that mathematicians and mathematics education researchers have in common -- a love of mathematics and a desire that more people (especially our students) learn to love, appreciate, and work with mathematical ideas flexibly. That said, there is also much that separates the two fields that could possibly lead to some misunderstandings of the aims and methods of those engaged in mathematics education research.

Having obtained a Ph.D. in mathematics and published several papers in my field (semigroups), I became increasingly interested in the problems my students were having. As a result, some fifteen years ago, I decided to take mathematics education at the undergraduate level as a serious research commitment. Consequently, I feel I have a "foot in both camps" and can understand and empathize with both. Indeed, many mathematicians I know seem to consider me a "math ed person" and a number of mathematics education researchers consider me a mathematician.

While much of what I have to say will be about research in mathematics education generally (i.e., at all levels, K-16+), many of the examples, and some of my remarks, will apply specifically to research in mathematics education at the undergraduate level, sometimes abbreviated as RUME.

---

[1] This paper was written in preparation for the author's AWM-MAA Invited Address at the MathFest in Burlington, Vermont, August 3, 2002. This address is given annually by the recipient of AWM's Louise Hay Award for Contributions to Mathematics Education. For more information on this award, see http://www.awm-math.org/hayaward.html.

[2] This statement has been attributed to Henry Pollak by Schoenfeld (2000, 2001).

What Mathematics Education Research Is and Is Not

A number of mathematics education researchers have published articles for the mathematics community describing the field; these have appeared in such places as *Notices of the AMS* and the *College Mathematics Journal* (Artigue, 1999, 2001; Schoenfeld, 1994, 2000, 2001; Selden & Selden, 1993, 2001; see also McKnight, Magid, Murphy, & McKnight, 2000, or Niss, 1999). These articles have provided inspiration for what follows, but the particular perspective expressed here is my own.

Describing, much less defining, the nature of mathematics education research is a daunting, virtually impossible, task. Indeed, a similar thing might be said about mathematics. When mathematicians were surveyed (Mura, 1993, 1995) and asked, "How would you define mathematics?", 33% replied, "I wouldn't't," a view I sympathize with. Still I will try to give some idea of the nature of research in mathematics education, the kinds of questions asked, and the kinds of answers, or partial answers, provided.

To describe a concept or idea, it is often best to provide both examples and nonexamples -- to say what it is and what it is not. I begin with the later.

What mathematics education research is not. It is not curriculum development per se, although it may involve developing some bits of curricula. It is not descriptions of interesting courses one has developed, although mathematics education research may involve the description of some teaching. Likewise, it is not writing a new textbook, developing an online course, or implementing a new way of teaching, although sometimes new/revised courses or pedagogies may come about as a result of research. It is not the development of novel assessment procedures, although novel/interesting tasks proposed by researchers are sometimes adapted for tests/examinations. It is not (local) evaluation studies that seek to answer questions like: Does our new algebra course really work? It is not institutional/departmental research yielding results such as: Almost everyone who fails our calculus readiness test and takes calculus anyway, fails it. It is also not interesting mathematics for use with one's students *if* one only had the time (e.g., *CMJ*'s Classroom Capsules). While these are all useful, and even scholarly, activities, they are not mathematics education research.

Mathematics education research is also not research about learning per se (with no consideration given to what is learned). To a large extent it concerns features specific to mathematics and different from those of interest in psychology such as the brief learning/memory of a string of nonsense symbols. So far, only a few general education, pedagogy, or psychology studies (e.g., Miller, 1956) seem to have been informative or useful in mathematics education research. One cannot assume, for example, that general cooperative group learning strategies will necessarily be appropriate for mathematics or for a specific content area within mathematics, such as conceptual difficulties with limit.

Mathematics education research is also not a collection of interesting anecdotes about student (mis-)conceptions or behavior vis-à-vis mathematical concepts. However, such informal observations can sometimes help one formulate interesting, researchable questions.

What is mathematics education research? It is *disciplined* inquiry[3] into the learning and teaching of mathematics, often involving close observations of students actively involved in challenging mathematical tasks. It is conducted using a variety of methodologies and it is *domain specific* -- it's about mathematics. Niss (1999), in an article explaining the nature of mathematics education research for mathematicians, described it as "the scientific and scholarly field of research and development which aims at *identifying*, *characterising*, and *understanding* phenomena and processes actually or potentially involved in the *teaching* and *learning of mathematics at any educational level*" (p. 5).

> Mathematics education researchers study how people learn and are taught mathematics, as well as the phenomena that influence teaching and learning. These include aspects of mathematics (calculus, functions, proofs), cognition (problem solving, misconceptions), psychological factors (motivation, affect, visualization), teaching methods (lecturing, cooperative learning, uses of technology and writing), change (individual teacher, institutional), and culture (gender, equity, the surrounding culture, cross-cultural comparisons). Research methods and background are borrowed from anthropology, sociology, and psychology, especially cognitive psychology, as well as from philosophy and artificial intelligence. (Selden & Selden, 1993, p. 432)

Since research in mathematics education is a social science, no single study can possibly hope to answer most useful questions definitively. Indeed, research in mathematics education "is a very different enterprise from research in mathematics . . . Findings are rarely definitive; they are usually suggestive. Evidence is not on the order of proof, but it is cumulative, moving towards conclusions that can be considered to be beyond a reasonable doubt." (Schoenfeld, 2000, p. 649).

For mathematics education research, one needs a solid understanding of the mathematics at, and beyond, the level at which the students being observed are working. In addition, mathematics education researchers, just like mathematicians, put a great deal of thought into selecting their research questions. They must be new, nontrivial, nonobvious, and the potential answer(s) should be interesting. In comparing mathematics and mathematics education research, Schoenfeld (1999) said, “The hard part of being a mathematician is not solving problems; it’s finding one that you can solve, and whose solution the mathematical community will deem sufficiently important to consider an advance. In any *real* research (in particular, education research), the bottleneck issue is that of problem identification – being able to focus on problems that are difficult and meaningful but on which progress can be made.”

The research question comes first. After it has been carefully formulated, one then selects appropriate data collection and analysis methods. If one wanted to know how many students continued on, and succeeded in, traditional mathematics courses after taking a reform college algebra or calculus course, one would use some quantitative methods, probably some simple statistics. On the other hand, if one wanted to know how students approached problems in an engineering mechanics course after taking a reform

[3] John Mason of the Open University has described research in mathematics education as *disciplined noticing*. For details, see Part V (pp. 151-216) of his new book (Mason, 2002).

calculus course or a traditional version, one would use a qualitative research method such as individual in-depth task-based interviews (Roddick, in press).

While researchers rarely write about how they decide on their data collection and analysis methods, occasionally one gains such insights from texts on research methods. For example, when Susan Pirie and Rolph Schwarzenberger wanted find out whether, and how, pupil-pupil discussion aided mathematical understanding, they considered a variety of data collection methods finally settling on audio-recording and taking notes -- from across the room -- as the least intrusive. For their analysis, after first considering ethnomethodology and conversational analysis, which focus on the organization of the talk, and sociolinguistics, which focuses on how social factors affect talk, they finally settled on discourse analysis as the most appropriate for the mathematical aspects of the pupils' talk (Pirie, 1998).

Some examples from my own work. In a series of three studies**,** John Selden and I investigated calculus students' problem-solving abilities. For the first study, we developed five moderately non-routine first calculus problems, administered them to paid volunteer C students as a test, complete with prizes for the best papers. The result, much to our surprise, was that not one student could solve a single problem completely, despite many of them currently being enrolled in Calculus II (Selden, Mason, & Selden, 1989). It then occurred to us that perhaps the students did not have the basic calculus skills needed to solve the non-routine problems. We thus devised a routine test consisting of precisely those calculus skills we considered necessary to solve our non-routine problems. Under similar volunteer testing conditions, we gave A and B students the same non-routine test, followed by this routine test. The most striking result was that only 9% of the problems were completely solved by these students, while having good calculus skills (Selden, Selden, & Mason, 1994).

We then decided to investigate the "folk theorem" that one really learns the mathematics of a given course in a subsequent course that uses it. Consequently, under similar conditions, we tested students who had taken all three semesters of calculus and were midway through a differential equations course. This time we got 14% completely correct solutions (Selden, Selden, Hauk, & Mason, 2000). We concluded that these students could not access their knowledge during problem solving. It may be that, when client disciplines, such as engineering, lament that many students cannot work applied problems, it is really the novelty that perplexes them.

In a more recent study, we investigated whether mathematics and secondary mathematics education majors at the beginning of a transition course could validate (i.e., check the correctness of) similar students' "proofs" of the elementary number theory theorem: *For any positive integer n, if* n^2 *is a multiple of 3, then n is a multiple of 3.* There were four such "proofs" -- one correct and three incorrect. The students were interviewed individually using "think aloud" protocols. Perhaps the most surprising result was that 46% of the time, these students could not correctly judge whether these purported "proofs" was indeed proofs -- a result that could have been obtained equally well by tossing a coin. At the end of the interview, when asked how they read proofs, these students indicated that the checked the logic, followed the reasoning step-by-step, and read for understanding -- several times even citing specific examples of how they had read a particular proof the night before in order to understand it (Selden & Selden, in press).

What Kinds of Questions Are Asked in Mathematics Education Research?

One can study an individual student, or pair of students, learning some piece of mathematics (a cognitive perspective). Or, one can study the interactions within a classroom, or within the broader school culture that do, or do not, promote the learning of mathematics (a socio-cultural perspective). One can also coordinate these two perspectives, trying to understand how both psychological and social factors are involved.

> In the case of individual cognition, one wants to know how students come to understand aspects of mathematics or how they develop effective mathematical practices, good problem-solving skills, or the ability to generate reasonable conjectures and to produce proofs. What goes on in students' minds as they grapple with mathematics and how might we influence that positively? More specifically, consider the difficulties that students have with the concept of limit. Does the everyday notion of speed limit as a bound present a cognitive obstacle? Does the early introduction of monotone increasing sequences constitute a didactic obstacle? Are there some, as yet neglected, everyday or school-level conceptions that university mathematics teachers might effectively build on? What is the influence of affect, ranging from beliefs through attitude to emotion, on effective mathematical practice? What roles do intrinsic and extrinsic motivational factors play?
>
> From a social perspective, whether of a single classroom or some broader community, one seeks information on how social interactions affect the group as well as the individuals involved. For example, how might one change the classroom culture so students came to view mathematics, not as passively received knowledge, but as actively constructed knowledge? Or, how might one restructure an entire curriculum to achieve this effect? What are the effects of various cooperative learning strategies on student learning? What kinds of interactions are most productive? Are some students advantaged while others are disadvantaged by the introduction of cooperative learning? Which students succeed in mathematics? Which students continue in mathematics and why? What is the effect of gender, race, or social class upon success in mathematics? In coordinating the psychological and social perspectives, any of the above questions might be asked, along with inquiry into the relationship between the two perspectives. For example, how does an individual's contribution affect a whole class discussion and conversely? (Selden & Selden, 2001, p. 237)

Answering the above kinds of questions requires labor-intensive, qualitative studies, usually with small numbers of students.

Yet, there are other interesting kinds of questions that might be asked and answered using more quantitative studies and relatively large numbers of students. There are enormous cross-national studies like the Third International Mathematics and Science Study, TIMSS, that can keep researchers busy for years (e.g., Schmidt, McKnight, & Raizen, 1996; see also http://ustimss.msu.edu/). But there are also useful questions regarding affect, beliefs, motivation, and other topics, that one can answer using questionnaire data. For example, to what extent do U.S. college students' values and beliefs about mathematics and mathematics learning resemble those of their instructors? One such study is currently being considered for publication.

Another kind of study, often referred to as a *meta-analysis*, considers data from a number of studies to see whether together they point to some consistent general conclusion. One such study asked: What causes talented students to opt out of mathematics majors? The answer seems to lie in the perceived poor quality of mathematics instruction as compared with that found in other disciplines (Linn & Kessel, 1996). Or, what is the effect of supplemental instruction (SI), a.k.a. help sessions, on students' grades and retention? One answer, after examining 177 courses in college algebra, calculus, and statistics from 45 institutions, concludes that SI students obtained better course grades than non-SI students and had a better chance of being able to continue on to the next mathematics course (Burmeister, Kenney, & Nice, 1996).

There are also *evaluation studies* that provide information about the effects of curriculum reform efforts on large numbers of students. For example, what are some longer-range effects on students of a given calculus reform project? During college, Project Calc students, on average, took one more mathematics course than traditional students (Bookman, 2001). For the C^4L Project, project students, on average, were found to take more calculus courses than traditional students (Schwingendorf, McCabe, & Kuhn, 2000). Such research tends to be generalizable, in the sense of providing detailed information about the students, the teaching, the project implementation, and the observed outcomes; this contrasts with institutional research providing purely local feedback to administrators, such as the percentage of remedial mathematics students who obtained a grade of C or better and went on to succeed in subsequent mathematics courses.

What One Can and Cannot (Reasonably) Expect From Mathematics Education Research

Mathematics education research, especially at the university level, is a relatively young field with an applied, or an applicable, character. Interesting questions, suitably modified for investigation, can arise when one encounters learning difficulties in one's classroom. For example, are remedial college algebra students handicapped because they do not understand the notation? Are real analysis students having difficulty because they can't generate suitable generic examples on which to build proofs? (Cf. Dahlberg and Housman, 1997; Rowland, 2002). Would drawing diagram help such students construct proofs? (Cf. Gibson, 1998.)

What research cannot tell us. Despite sometimes arising from everyday questions that may occur during teaching, mathematics education research does not usually provide immediately applicable prescriptive teaching information, rather it provides general guidance and hints that might help with teaching and curriculum design. While mathematics education research may inform us, it cannot tell us which curricula or pedagogies are "best" because such decisions necessarily involve value judgments about what is important for students to know (Hiebert, 1999). For example, if engineering students use computer algebra systems such as *Mathematica* or *Maple*, research cannot tell us which calculus and matrix algebra computations students should perform flawlessly by hand. It might, however, advise us on the extent to which our chosen curricula, once implemented, have succeeded in attaining our goals.

What research can tell us. Mathematics education research does provide empirical results about what students and teachers do, or do not, know and understand. It can include the design, implementation, and study of the effects of specially designed curricula; French researchers refer to this as *didactic engineering*. (A sampling of some of these empirical results is given below). More recently, there have been studies of how mathematics is used in the workplace by automobile workers, nurses, bankers, biologists, and other scientists (Smith, 1999; Hoyles, Noss, & Pozzi, 2001; Noss & Hoyles, 1996; Smith, Haarer, & Confrey, 1997; Roth, 1999).

However, perhaps more importantly, mathematics education research provides us with ideas/concepts and the corresponding words to talk with. I am *not* referring to unnecessary jargon, but rather to new conceptualizations and new metaphors for thinking about and observing mathematical behavior. It is very difficult to notice patterns of behavior or thought without having appropriate concepts. One needs a lens with which to focus, or frame, what one is seeing. (Cf. Selden & Selden, 2001).

Concepts to Think and Talk With

Below are a few examples of concepts that researchers in mathematics education have found useful; they have been used to help formulate research questions, select research methodologies, analyze data, and draw conclusions. They also provide us words with which to think and to conceptualize phenomena.

Concept definition versus concept image. We all have experienced the mismatch between concepts as stated in definitions and as interpreted by our students.

> The terms ***concept definition*** and ***concept image*** distinguish between a formal mathematical definition and a person's ideas about a particular mathematical concept, such as function. An individual's concept image is a mental structure consisting of all of the examples, non-examples, facts, and relationships, etc., that he or she associates with a concept. It need not, but might, include the formal mathematical definition and appears to play a major role in cognition. (See Tall and Vinner, 1981.) Furthermore, while contemplating a particular mathematical problem, it might be that only a portion of one's concept image, called the ***evoked concept image***, is activated. These ideas make it easier to understand and notice various aspects of a student's thinking, for example, to understand the thinking of a student who conceives of functions mainly graphically or mainly algebraically without much recourse to the formal definition. (Selden & Selden, 2001, p. 240)

Students' knowledge about a concept, like limit, is often compartmentalized with the formal definition perhaps only loosely connected with other facts and examples about that concept. This explains why students can often quote the formal definition, more or less accurately, when asked to do so, but fail to use it in other appropriate situations.

***Epistemological,*[4] *cognitive, and didactic obstacles*.** Another set of ideas that has proved to have powerful explanatory power is that of various kinds of obstacles.

> When applied to the learning of mathematics, these refer, respectively, to obstacles that arise from the nature of particular aspects of mathematical knowledge, from an individual's cognition about particular mathematical topics, or from particular features of the mathematics teaching. An obstacle is a piece of, not a lack of, knowledge, which produces appropriate responses within a frequently experienced, but limited context, and is not generalizable beyond it (Brousseau, 1997). Using the historical development of function as a guide, it has been found that one epistemological obstacle that students need to overcome is the idea of function as expression, just as was the case with Euler (Sierpinska, 1992). (Selden & Selden, 2001, p. 240-241).

In making the transition from high school to university mathematics, students frequently encounter new ways of conceptualizing previously well-known concepts -- ways that can conflict with well-honed prior mathematical practices -- causing them considerable difficulty. Such didactic obstacles, often require students to make quite difficult reconstructions of their mathematical knowledge. For example, in high school geometry, the tangent line to a circle is often defined to be that unique straight line that touches the circle at just one point and is perpendicular to the radius at the point of contact. However, upon coming to calculus, the tangent to a function at a point is defined as the limit of approximating secant lines, and somewhat later, as the line whose slope is given by the value of the derivative at that point. Research in France by Castela (as reported in Artigue, 1992, p. 209-210) found that many of the 372 secondary students questioned, after having studied analysis (i.e., calculus) for a year, had great difficulty determining from a graph whether a given line was tangent to a particular function at an inflection point or a cusp.

Another example is provided by the treatment of equality in analysis. Whereas in secondary school algebra and trigonometry, students have grown used to proving that two expressions are equal by transforming one into another using known equivalences, in analysis one can prove two numbers a and b are equal by showing that for every $e > 0$, one has $|a-b| < e$. That this idea is not an easy one is indicated by a French study in which more than 40% of entering university students thought that if two numbers A and B are less than $1/N$ for every positive integer N, then they are not equal, only infinitely close (Artigue, 1999, p. 1379). Indeed, various notions related to the concept of infinity provide examples of epistemological obstacles.

Didactic contract. Students often come to classroom situations with well-entrenched sets of (often tacit) expectations about how the teaching and learning will take place -- about the respective roles of the teacher and the students vis-à-vis the subject. This seems to be particularly so in mathematics. Brousseau (1997) described this by saying that everything happens *as if* there were a contract between the

[4] The idea of epistemological obstacle was introduced by G. Bachelard, and subsequently G. Brousseau brought this idea into mathematics education research via his theory of didactical situations (1997). B. Cornu, and somewhat later A. Sierpinska, analyzed epistemological obstacles in the development of the limit concept and linked these to students' difficulties.

mathematics teacher and the mathematics students and referred to this as the *didactic contract*.[5] In U.S. mathematics classes, it is often the case that, and hence students expect that, the teacher will first invite students' questions, which will be primarily of the sort, "How do you work problem #7?" Only after this will the teacher introduce the new material and work some sample problems before finally assigning similar homework exercises. This seems to be a well established tradition; at least as early as the eighth grade, students expect this pattern. The teacher is also expected to be an authority who delivers well-laid out lectures which contain copious detailed sample solutions to model problems, answers all questions (preferably, briefly), and clearly delineates her/his expectations of students. While students find this dull, they also find it comforting.

Any attempt to change this didactic contract without adequate initial explanation and preparation by the teacher -- referred to as *renegotiating the didactic contract* -- can result in student unease, complaints, or even outright rebellion. Even when explicit attention is given to renegotiation of the didactic contract, students can feel and express considerable discomfort. This happened in a number of reform calculus courses in the U.S. (e.g., Bookman & Friedman, 1994, p. 114). It also happened in the Warwick Analysis Project, an experimental first-year university analysis course in which students considered a carefully structured sequence of questions that developed the rationale for the main definitions, constructed the central arguments behind the main theorems, and used those theorems in subsequent arguments. One student said, “What I would really like is if we could have a lecture, and then be given a set of question based on the lecture, and do it in class.” (Alcock & Simpson, 2001, p. 104)

To renegotiate the didactic contract takes effort on the part of the teacher. In the *method of scientific debate*, a way of teaching calculus/analysis to first-year French university students that emphasizes conjecture, discussion, and argument and culminates in proof, this new contract is clearly delineated for the students in the first few class sessions. In addition, the teacher must continue to act in accordance with the contract and insist that students do so, too. (Cf. Tall, 1991, pp. 136-7, 224-229.)

In the process of renegotiating the didactic contract, one may want to consider the related, but independently developed, ideas of classroom social norms and sociomathematical norms. *Social norms* refer to "those aspects of classroom social interactions that become normative." Every class develops these -- whether students are expected to sit quietly and take notes or that they should explain their thinking, offer alternative explanations, and make sense of other students' reasoning. *Sociomathematical norms* are related specifically to mathematics. In a traditional class, a sufficient mathematical justification in response to "How did you get that?" might be "That's the rule." However, to promote the idea that explanations have to be mathematical, a teacher might respond to a student's proposed interpretation of a differential equation by saying, "Tell us why you made that conclusion" (Yackel, Rasmussen, & King, 2000).

Process-object duality. There are many concepts in mathematics, such as function, that can be viewed both as processes -- as having calculational or procedural aspects -- or as objects -- as entities that can be acted upon, perhaps by other processes. Often, the process, or its precursor in the form of some concrete action, is taught and experienced first. For example, function might first be encountered via concrete actions,

---

[5] The notion of a contract to describe and analyze the interactions of people goes back at least to Jean-Jacques Rousseau who wrote about the *social contract*.

for example, by computing x^2+1 for some specific numbers, thought of as "first square the number, then add 1." Subsequently, it may become possible to think in terms of general inputs and outputs, without invoking a particular algorithm. Later, functions may come to be seen as objects in their own right, e.g., as things which can be acted upon, say by a differential operator. A great many individual concepts such as limits, sequences, cosets, and quotient groups, have been investigated using an *action-process-object-schema* (APOS) view (see Dubinsky & McDonald, 2001). Somewhat similarly, Sfard (1991) has described an individual's journey from an *operational* (process) to a *structural* (object) conception as reification.

Often, a single mathematical notation is used to designate both process and object conceptions. For example, $5+4x$ stands for the process of adding 5 to the product of 4 and x, as well as the result of that process. In order to be able to deal with mathematics flexibly, students need both the process and object views of many concepts, as well as the ability to move between the two views when appropriate. Concepts that can be viewed both as processes and objects are sometimes called *procepts* (Tall, 1991). Students need our help -- in order to be aware of both process and object views and in navigating successfully between them.

Differences and Similarities Between Research in Mathematics and Mathematics Education

The following is a highly personal and eclectic perspective, detailing various aspects of being in, and doing, research in both mathematics and mathematics education. Since the most noticeable difference seems to be that there are no proofs in mathematics education, I will begin there.

The roles of proofs, evidence, conjectures, and definitions. Mathematicians specialize in proofs, long deductive arguments, that establish the truth of theorems (mathematical facts). Mathematics education researchers rarely make long deductive arguments, rather they look for regularities in the behaviors they observe and report their observations and their conclusions. These conclusions are necessarily tentative; they are based on data collected using a variety of suitable methods. Since the evidence (data) is empirical, no single study can be conclusive. One is looking for compelling evidence. Even for a single study, whenever possible, data from several sources, such as one's own field notes, audio- or videotapes, and interviews of participants, are compared to see whether they all point to the same, or similar, conclusions -- a process known as *triangulation*. (For more details, see Schoenfeld, 2000, p. 648 or Schoenfeld, 2001, p. 234.)

> Results in mathematics education calls (*sic*) for synthesis and interpretation in ways unlike mathematics. A mathematician, before using a result, typically does no more than check its proof. Unless an error is discovered, a result true one year is true the next. In mathematics education research, the situation is less clear-cut. Even very careful observations only suggest, but do not prove, general principles. Corroboration by other studies is important. The discovery of additional information can make what previously appeared firmly established, less so later. (Selden & Selden, 1993, p. 432)

Mathematicians, perhaps after some initial observations, conjectures, and experimenting with what seems interesting or important, make formal if-and-only-if definitions that stabilize the concepts they investigate. They work within formal systems, whose axioms may have been inspired by the consideration of real world problems; however, the results they deduce may, or may not, turn out to be applicable. Mathematics education researchers, on the other hand, investigate and seek to describe what's out there -- the learning/teaching of mathematics by actual students and teachers. That means they are dealing with things in the world, and consequently, the definitions they use are necessarily descriptive.

> Definitions [in mathematics education research] do not have, perhaps cannot have, mathematical precision. (Try catching the meaning of "understanding" exactly.) Redundancy can be useful, whereas in mathematics, it is usually avoided, possibly because it impedes the comprehension of complex proofs. Perhaps because of such differences, those unfamiliar with education research occasionally dismiss as "jargon" the introduction of concepts not easily expressible with mathematical precision. (Selden & Selden, 1993, p. 432)

Issues involved in doing research and getting it published. Mathematicians rarely have to consider differing philosophical/theoretical perspectives. While techniques of proof may vary from one subfield to another, with proofs in analysis looking quite different from those of algebraic topology, still there is a "gold standard" and it is proof. By contrast, in mathematics education research a single situation can be analyzed from several different philosophical/theoretical perspectives. For example, a person's knowledge can be viewed as something, perhaps ever changing, that exists "in the head." Such knowledge might be seen as individually developed or as a socially and culturally mediated, taking either a Piagetian or a Vygotskian stance. (Cf. Selden & Selden, November 2001.) The mind might even be viewed as some sort of information processing device. Or, one can view knowledge as situated, that is, as consisting of how an individual interacts with, or functions in, various situations and consider learning to be a kind of cognitive apprenticeship (Lave & Wenger, 1991).

Mathematicians do not need to worry about expenses associated with collecting data. Nor do they need to be concerned about ethical considerations making sure that the research does no harm to participating students and teachers and that confidentiality is maintained. Along with this goes the hassle of satisfying the university's human subjects committee or institutional review board that sometimes seems more intent on assuring that the university not be sued, than on helping the research conduct a careful, confidential, and ethical study.

Undergraduate and graduate students in mathematics learn, or do not learn, to prove theorems by taking courses and seminars in which they are asked to prove theorems. As undergraduates, perhaps they have taken one "transition to proof" course. They are not often asked to make reasonable, interesting conjectures. Mathematics education graduate students, on the other hand, often take at least one course in research methodology, covering various qualitative and quantitative methods. They may also conduct small pilot studies in other courses. They may have some practice, but probably not enough, in formulating interesting, manageable, research questions.

Furthermore, the way one presents one's research for publication, in mathematics and mathematics education, also has similarities and differences. It seems pretty clear-cut in mathematics whether one has established what one has claimed -- either one has presented an error-free proof or one hasn't. This is not to say that refereeing a mathematics manuscript is easy, but only that value judgments of a manuscript's worth consist of whether a result is nontrivial, nonobvious, new, and interesting to a significant number of a journal's readers.

For mathematics education research papers, on the other hand, in addition to asking whether the authors have addressed an interesting question, there are other considerations. What theoretical framework have the authors used? Is it appropriate for the study at hand? Have the authors chosen an appropriate methodology? Have they described it in sufficient detail that readers can tell the study was carefully executed? What claims are they making? What evidence do they marshal in support of these claims? Is that evidence appropriate? Is the data presented in a clear and understandable way? Have the authors drawn appropriate inferences? Do they indicate, via a literature review, how their work relates to that of others on the same topic? If they give implications for teaching, are these based upon their research and/or other results found in the literature? If it's a quantitative study, are the statistical methods used appropriate? If it's a qualitative study, does it provide "thick descriptions" of the students and what was observed? Is enough information and detail given so the study is replicable by others? (Cf. Hanna, 1998; Lester & Lambdin, 1998.)

It might seem that some of these questions are matters of common sense and unlikely to be answered negatively. But my own experience on the Editorial Panel for the *Journal for Research in Mathematics Education*, reading some 15-20 manuscripts per year, suggests any of them can have a negative answer. Perhaps because the nature of mathematics education research is not well understood, and sometimes the boundary between what is and what is not such research is somewhat fuzzy, a number of people -- mathematics teachers, mathematicians, masters students in education -- submit their interesting ideas for classroom use or poorly thought-out studies to mathematics education research journals.

In addition, one usually needs more reviewers (referees) for a mathematics education research manuscript -- usually, three to five, with four quite common. Editors need advice on several aspects of a manuscript, especially for those which are far removed from their own expertise. A given paper might need reviewers with expertise in discourse analysis, socio-cultural perspectives, semiotics, or ethnography, or if there is significant quantitative data, in statistics -- not to mention mathematics itself. Reviewers (referees) not only comment on what is wrong with a paper, but also make helpful suggestions for its improvement, especially if the recommendation is "revise, resubmit, re-review." If a manuscript is not research, but contains useful ideas for teaching, reviewers generally try to suggest publications that do have an interest in such work, e.g., *The Mathematics Teacher* or *PRIMUS*. If the research addresses an interesting question, but a reviewer finds problems with the methodology, he/she may advise using the work as a pilot study or make suggestions for further investigation.

The emphasis seems to be on ensuring that only high quality work gets published, while at the same time encouraging the development of researchers which are in short supply. This contrasts with my own experience with referees in mathematics; I found comments on my early papers to be direct and to-the-point, but occasionally needlessly acerbic. Instead of feeling welcomed as a potential future colleague, a novice researcher might well interpret such remarks as saying, "If you can't cut it, get out." It seems that such attitudes towards mentoring are not dead yet in mathematics departments at some research universities. Recently, the chair of Berkeley's mathematics department said, "One of our goals is to cultivate self-reliance. Berkeley is a tough place. Berkeley is not a warm and fuzzy place. Students react to this atmosphere: Some thrive, and others don't."[6]

Public perceptions of the two fields. Mathematics, and consequently research in mathematics, has the advantage of being considered hard and potentially useful to science and technology; ergo to the economy. Furthermore, the history of mathematics goes back thousands of years, giving it a well-established pedigree. Mathematics education research, on the other hand, is a relatively young field, even if one goes back to work in psychology by Thorndike (1922). Furthermore, much of that early research, done in the connectionist and behaviorist traditions of psychology, is in what is today referred to as the “agricultural model” in which one class of students, just like one field of corn, is given some treatment, while another class is considered the control group. Such studies, which rarely concerned what was in students' minds, have not been found to be very useful. It is simply not as easy to control the variables when students, rather than corn plants, are being considered. Thus, the current foundations of mathematics education research can really only be traced back about as far as the cognitive science revolution of the 60's, or even somewhat later. This lack of a long history and the variety of existing research paradigms may contribute to confusion about what is and is not research. And while mathematicians are quite often held in awe (perhaps for the wrong reasons), mathematics education researchers sometimes feel their work is misunderstood, if not denigrated. It is certainly no help when some on both sides of the "math wars" call on differing bits of research to support entrenched political points of view about the teaching and learning of school mathematics.

Some positives and negatives of being a mathematics education researcher. Mathematics education researchers seem to have exceptionally high service demands put on them. Of course, both mathematicians and mathematics educators are called on for their share of committee work and student advising. Beyond that, mathematicians may occasionally be called on to do such things as coach the Putnam Prize team or assist with school mathematics contests. Mathematics educators, while being much fewer in number in a department of mathematics, are often called upon to train teaching assistants, conduct workshops for teachers, supervise student teachers, and apply for education grants (that almost always have a large component of mostly service, with a little research "around the edges").

Despite a few negatives to becoming a mathematics education researcher, such as greater service demands, there are a great many positives. If one is doing research about undergraduates' learning, one can find many interesting, potentially researchable, questions arising out of one's teaching. If one listens carefully and wonders why are students doing or saying something, and if one further assumes that they are genuinely

---

[6] This remark was made at an NSF VIGRE workshop after a third-year review caused Berkeley to lose its grant (Mackenzie, 2002, p. 1390).

engaged in sense-making, then one may be led to a researchable question, especially as so much research remains open at the undergraduate level. This is not to say that a few casual observations *are* research, but that appropriately reformulated, they can lead to conjectures and research questions. Contrast this with mathematics. It is rare, except perhaps at a major research university, to have conjectures or research ideas arise directly out of one's interaction with students.

Another positive is the number of currently available, and perhaps soon-to-be available jobs, in academia. A survey of 48 institutions with doctoral programs in mathematics education, conducted in 1999, reported that almost 80% of then-current faculty would be eligible for retirement in the next ten years, with 38 (of the 48) anticipating hiring a total of approximately 75 faculty members in the next five years (Reys, Glasgow, Ragan, & Simms, 2001). This does not include the many other opportunities for mathematics education researchers, in mathematics and education departments, at other colleges and universities in the U.S. Freshly-minted mathematics education PhD's of my acquaintance have gotten as many as half a dozen interviews at top colleges and universities, whereas a new mathematics PhD may consider himself lucky to get one or two. The job prospects for mathematicians in academia, although getting better, are hardly excellent. This "acute shortage" of mathematics education PhD's was highlighted in a recent *AMS Notices* article -- in each of the previous two years, there were more than 300 advertised openings in mathematics education. At the University of Missouri-Columbia in the year 2000, for eight openings in mathematics, including postdocs, there were over 400 applicants, whereas for three tenure-track positions in mathematics education, there were fewer than twenty applicants (Reys, 2001).

The Kinds of Claims Made and Evidence Provided

In reports of quantitative studies, it is important that the researcher indicate explicitly what indicator(s) have been used to measure a particular construct such as student knowledge. For example, test, examination, or course grades, a national test such as the NAEP or NELS,[7] a researcher-designed test/questionnaire, or some combination thereof might be used as indicators of student knowledge. Researchers, and readers, must use caution when interpreting correlations between proxy measures for such concepts as student knowledge and teacher knowledge.

For example, in the 70's several studies of the relationship between student learning and teacher knowledge took as the indicator of teacher knowledge the number of university-level mathematics courses successfully completed; using this indicator, no

[7]The National Assessment of Educational Progress (NAEP), also known as "The Nation's Report Card," is mandated by Congress and conducted by Educational Testing Service biennially to provide a "snapshot" of what U.S. students in grades 4, 8, and 12 know and can do. The National Education Longitudinal Study (NELS:88) was conducted for the National Center of Educational Statistics, part of the U.S. Department of Education, in 1988. A 1 1/2 hour multiple-choice test, covering reading, history, mathematics, and science, was given to thousands of 8th, 10th, and 12th graders across the country; in addition, extensive questionnaire responses were gathered from students, their parents, the school, and two of each student's teachers.

relationship was found. In these studies, no attempt was made to measure what the teachers actually knew, for example, by a research-designed test or task-based interviews. At the time, Begle concluded, somewhat too hastily in retrospect, that "the effects of a teacher's subject matter knowledge . . . seem to be far *less* (emphasis added) powerful than most of us realized. . . . Our attempts to improve mathematics education would *not* profit from further studies of teachers." (School Mathematics Study Group, NLSMA Report, 1972; Begle, 1979; as cited in Fennema & Franke, 1992, p. 148)

Yet, subsequent studies of teachers' mathematical knowledge have proved informative. For example, an in-depth report of one senior preservice K-8 teacher who had completed her first three years as a mathematics major, showed that despite wanting to teach for both procedural and conceptual knowledge, she could not give conceptual explanations for common middle school topics, e.g., she had no conceptual explanation of pi and did not know its relevance to the circumference and area of a circle (Eisenhart, Borko, Underhill, Brown, Jones, & Agard, 1993). Thus, it would seem that successful completion of even a number of advanced college mathematics courses is not sufficient. What is lacking? Leping Ma's (1999) in-depth comparative study of 23 U.S. and 72 Chinese elementary school teachers reported that while both could do and explain procedures, the vast majority of the Chinese teachers had a thorough conceptual understanding of such topics as place value, whereas many of the U.S. teachers lacked this. Many other earlier studies (e.g., Ball, 1989; Lampert, 1990) had also found a lack of conceptual understanding.[8]

What inferences might we draw? One possibility is that undergraduate mathematics courses aren't addressing the kinds of mathematical understanding K-8 teachers need. Indeed, this is one of the conclusions of the National Research Council Study, *Adding It Up: Helping Children Learn Mathematics* (Kilpatrick, Swafford, & Findell, 2001). For details, see chapter 10, "Developing Proficiency in Teaching Mathematics."

For both qualitative and quantitative research it is important for researchers to explicitly describe their background assumptions, theoretical perspectives, the kinds of research questions asked, their data collection and analysis methods, and to make a clear argument that links their data to their conclusions. (Cf: Lester and Wiliam, 2000). In addition, in order for qualitative research with just a few students to be generalizable, the researcher should provide "thick" descriptions of the students, the classroom situation, the teaching, etc., so readers can judge for themselves whether their situations are sufficiently similar to that of the researcher so as to use the results meaningfully. Ultimately it is up to readers to determine whether the evidence presented in support of a claim is convincing. Of course, if an article is published in a reputable journal, a reader has the assurance that at least the editor(s) and several reviewers have found the question interesting, the claims made reasonable, and the evidence marshaled in support of them convincing. (See below, for a sampling of questions reviewers ask.)

---

[8] In a phenomenon inexplicable to some mathematics education researchers including one journal editor of my acquaintance, the study reported in Leping Ma's (1999) book seems to have resonated with, and disturbed, many U.S. mathematicians. Whereas prior research studies reporting similar findings remain virtually unknown to mathematicians, Ma's book has elicited numerous mentions and several reviews by mathematicians (e.g., Howe, 1999), as well as an AMS news release (dated August 12, 1999) referring to her study as "groundbreaking."

It is sometimes said that qualitative research is mainly descriptive, that while it can be useful for building theory, uncovering students' (mis-)conceptions, and suggesting interventions, it cannot identify generalizable strategies for academic success. Critics of education research sometimes suggest that only rigorous controlled experimental, or quasi-experimental, field research, often in the form of control group vs. experimental group studies, should be used especially in making policy or curriculum decisions (Carnine & Gersten, 2000). They often look to medical research as the sine qua non, and as a consequence, tend to label educational research as unscientific, especially when it is qualitative, descriptive, and based on small numbers of students. However, to a large extent, studies which begin with one, or more, null hypotheses and students (somewhat) randomly assigned to either the treatment or the control group, have been found uninformative and have been given up within the mathematics education research community (Schoenfeld, 1994; Kilpatrick, 1992). Furthermore, even in traditional scientific disciplines, information can be gained by considering just a few cases. For example, recently an entirely new insect order -- the first since 1914 -- was named and described based on just two extant museum specimens (Klass, Zompro, Kristensen, & Adis, 2002). Also in cognitive neuropsychology, studies of just one or two patients with specific brain lesions can provide information on functions of small regions of the cortex. (See for example, Dehaene, 1999, ch. 7.)

Furthermore, mathematics education research is not meant to be a "hard science" in the sense of physics. Results are often suggestive, and rarely predictive. However, information and insights are gained on a wide variety of questions using a variety of research methods and theoretical perspectives. Data are collected and analyzed, evidence is provided, and arguments linking claims to that evidence are given. Corroboration of results by subsequent studies is important. "Nevertheless, research in mathematics education shares a powerful ‘pyramiding’ characteristic with other sciences; results rely on careful observations, are separated from investigators' opinions, and are subject to community scrutiny, so that subsequent work can be based on them (Selden & Selden, 2001, p. 238).

A Sampling of Results from Research in Mathematics Education

The following is a potpourri of recent results, mostly from research in collegiate mathematics education. For topics such as functions, variable, calculus, linear algebra, proof, problem solving, and assessment, it ranges over students' misconceptions, their ways of thinking and working, their difficulties, the time needed to become flexible knowers and users of mathematics, the effects of reform versus traditional calculus, and the influences of beliefs, everyday language, gender, and the surrounding culture on the teaching and learning of mathematics.

Variables and functions. While secondary students' conceptions, and misconceptions, in algebra have been quite well studied (see Kieran, 1992), less is known about beginning college students' conceptions. University mathematics courses usually assume a good knowledge of high school algebra, in particular, an understanding of variable as a general number, as a specific unknown, and as an indicator of a variable quantity in a functional relationship. While all these uses of variable are included in the secondary school curriculum, a study at a private Mexico university suggests these distinctions are not clear to many incoming students (Trigueros & Ursini, in press).

Perhaps because functions play a central and unifying role in mathematics, there have been many studies on what students do, and do not, know about functions at various stages. For example, beginning college students often have limited concept images that include various polynomial examples, but usually not the constant function, as well as a few other functions, such as log, exponential, and trig, that they have studied in class.

> Research from France, England, Israel, Poland, and the United States has revealed numerous common student misconceptions. Functions are seen as rules with regularities; this is the "function as formula" idea. Functions are often identified with just one representation -- either the symbolic or the graphical. A change in the independent variable is seen as causing a change in the dependent variable, with the consequence that constant functions are often *not* considered functions. The vertical line test is used almost exclusively in determining whether a given example is a function. Even functions expressed via polar coordinates are tested this way! (Selden & Selden, 1993, p. 439)

Furthermore, it seems to take even our best students until early in their graduate school careers to exhibit a flexible knowledge of function, especially when confronted with an unfamiliar problem. Carlson (1998) found this when she interviewed students who had just received A's in college algebra, second-semester honors calculus, and first-year graduate mathematics courses. (For more on functions, see Harel & Dubinsky, 1992; Leinhardt, Zaslavsky, & Stein, 1990; or Thompson, 1994.)

Calculus. Results on student learning of calculus abound. For example, Williams (1991) found U.S. college calculus students often had a dynamic view of limit in terms of "approaching" which they were reluctant to give up, despite his attempts to have them adopt a more formal view of limit by presenting them with situations to engender cognitive conflict. Although Williams did use the concept definition/concept image distinction, he saw students' resistance to change as influenced by their prior experiences with graphs of simple functions, the value they put on simple techniques for getting answers, and on the tendency to view anomalous problems as minor exceptions to the rules.

In a study about students' sources of conviction, college calculus students with external sources of conviction, such as the teacher or text, viewed calculus as a collection of facts and procedures to be memorized, claimed they neither understood nor valued the underlying theory, and had misconceptions of limit as bound or unreachable. In contrast, those with internal sources of conviction, such as appeals to empirical evidence, intuition, logic, or consistency, thought mathematics was supposed to make sense, that calculus was logical and consistent, expressed frustration when told in class to use a formula but not why it works, and paid a lot of attention to how formulas are derived and how theorems are proved (Szydlik, 2000).

Students with at least two semesters of single-variable calculus were observed as they sketched the graphs of functions given information about their first and second derivatives, limits, and continuity; they were also interviewed. The students tended to examine one interval at a time, using mostly facts about the first derivative, and found it difficult to put together information across contiguous intervals. Baker, Cooley, and

Trigueros (2000) concluded these students had two distinct, but interrelated, schema -- one for properties, the other for intervals -- which they had difficulty coordinating.

After teaching and studying a conceptual calculus course for first-year Australian university students, White and Mitchelmore (1996) conjectured that a major source of students' difficulties in applying calculus lies in an underdeveloped concept of variable -- variables are treated as symbols to be manipulated rather than as quantities to be related. They suggested that entrance requirements for calculus courses be made more stringent in terms of variable understanding or an appropriate precalculus course be offered at university level. Another paper reports on the effects on subsequent mathematics, physics, and engineering courses of having three versions of calculus -- traditional, Harvard, Calculus using *Mathematica* -- simultaneously available to students in both small and large sections. No significant difference in grades was found, regardless of the variety of calculus studied. Even allowing for switching between sections, the authors believe "math departments are justified in teaching traditional and reformed calculus concurrently" (Armstrong & Hendrix, 1999).

Everyday knowledge and intuition can get in the way. Courses for preservice teachers often include a some beginning concepts on sets and infinity. These students often think that "is an element of" is transitive; perhaps using their everyday experience with "in," wherein if one has an aspirin in one's cosmetic bag inside one's purse, one normally says one has an aspirin in one's purse (Zazkis & Gunn, 1997). To make matters worse, their teachers may use "in" for both "is an element of" and "contained in."

Using their prior experiences and intuitions, naive students generally use one, or more, of three criteria -- inclusion, single infinity (i.e., all infinite sets have the same cardinality), and 1-1 correspondence -- to judge the size of infinite sets. In Israel, where preservice secondary teachers routinely take a year-long traditional course on set theory, the 1-1 correspondence criteria is presented as the method for determining cardinality, often with total disregard for these students' pre-existing intuitions regarding infinity. When interviewed after the course, students thought 1-1 correspondence was suitable for comparing the cardinality of infinite sets, but this was not accompanied by an increased tendency to reject other criteria, illustrating once again how hard it is to dislodge strongly held incorrect intuitions (Tsamir, in press).

There are many more examples of recent research in undergraduate mathematics education, including work on preservice teachers' beliefs and knowledge, on linear algebra, the influence of language, and assessment. (For a compact summary of somewhat older results, see Selden & Selden, 1993, pp. 436-439.)

What RUME Might Have to Offer University Teachers of Mathematics

While research on undergraduate mathematics education is growing, it is a young field and much more needs to be done. Thus far, work has concentrated mainly on a few areas, such as function, calculus, and pre-service teachers, that overlap with research at the pre-university level. Much less work has been done on undergraduate students' attitudes, beliefs, and motivation; their understandings of, and difficulties with, concepts from other content areas such as linear algebra, modern algebra, and real analysis; their understandings and use of formal definitions; their approaches to nonroutine problem solving and proof; on teaching strategies that work; or on university teachers attitudes, beliefs, and values.

Still, some hints and suggestions for teaching can be gleaned from existing research, with more anticipated when more is known. As Artigue has said,

> . . . research carried out at the university level helps us to understand better the difficulties in learning that our students have to face, the surprising resistance to solutions of some of these difficulties, and the limits and dysfunction of our teaching practices. In various cases research has led to the development of teaching designs that have proved to be effective, at least in experimental environments. (Artigue, 1999, p. 1384)

Studies make us aware of which mathematical concepts are students are likely to find particularly difficult, and sometimes even why. Sometimes it is everyday knowledge, such as limit as a bound, that gets in the way of learning new concepts. Other times, it is earlier (tacitly assumed) techniques or conventions, such as how one establishes equality of two objects, that provide an obstacle to new learning.

One thing seems pretty certain, there are no easy fixes to some of these problems. While we would like inexpensive and easy solutions, most of the time these do not exist. Most reform, or experimental, courses require significant changes and more effort on both the teachers' and students' parts.

> One essential reason is that the problems are not only with the content of the teaching (it is not enough to write or adopt new textbooks); the problems are related to the forms of students' work, the modes of interaction between teachers and students, and the forms and contents of students' assessment. Changes are not so easy to achieve; they require time and institutional support, and they are not merely a matter of personal good will. (Artigue, 1999, p. 1385)

However, I feel some small innovations, whether these be the use of writing prompts or occasional group work, initiated by individual teachers can bring about worthwhile learning experiences for students.

There is other work on cognitive conflict teaching, on problems/tasks from research that one might use in one's teaching or assessment of students, on the care that needs to be exercised when implementing technology in teaching, and on being careful not to "dumb-down" curricula, especial problems and projects.

What I'd Like to See Happen: Two Suggestions

In closing, I'd like to make two suggestions for our community to consider. The first comes from last year's AWM Louise Hay Awardee, Pat Shure, who made an excellent suggestion in her invited address to this group. I want to reiterate it very emphatically. Pat suggested that

> . . . we advocate for a Mathematical Education Research Institute [MERI] modeled on the same principles as MSRI [Mathematical Sciences Research Institute] whose funding comes primarily from NSF with additional support

> from other government agencies, private foundations, and academic and commercial sponsors. The Institute would run programs to involve researchers from all of the overlapping areas of mathematics education." (Shure, 2001)

Finally, I'd like to put forth a suggestion that John and I have been thinking about for a long time. We know a number of excellent, dedicated university teachers of mathematics who put a great deal of time and thought into developing courses, especially single-section upper-division courses where there is great freedom to innovate. Often these are very successful. However, knowledge of them disappears when the innovator no longer teaches the course. Such courses should be written up and preserved so they can be replicated and others can benefit -- either in a print or on-line journal. Our suggestion is that the mathematician team up with a mathematics education researcher to provide a detailed description of the course, explaining the ideas behind the innovation, what new/different pedagogies were used, what materials were found to be effective, what assignments/projects were undertaken by the students with what results. But most important of all, the authors should provide informed conjectures about why the course was effective, using mathematics education research theory and results so informed modifications can be made. Although not their main purpose, such articles would provide a published, and refereed, record of the "scholarship of teaching" for tenure/promotion.

Thanks for listening.

References

Alcock, L. & Simpson, A. (2001). The Warwick Analysis Project: Practice and theory. In D. Holton (Ed.), *The teaching and learning of mathematics at university level: An ICMI study* (pp. 99-111). Dordrecht, The Netherlands: Kluwer Academic Publishers.

American Mathematical Society. (1999, August 12). *Understanding of fundamental mathematics key to improving mathematics education* (News Release). Retrieved June 19, 2002, from American Mathematical Society Online New-in-Math Online Page: http://www.ams.org/new-in-math/press/under.html.

Armstrong, G. M., & Hendrix, L. J. (1999). Does traditional or reformed calculus prepare students better for subsequent courses? A preliminary study. *Journal of Computers in Mathematics and Science Teaching, 18*, 95-103.

Artigue, M. (1992). The importance and limits of epistemological work in didactics. In W. Geeslin & K. Graham (Eds.), *Proceedings of the sixteenth conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 195-216). Durham, NH: University of New Hampshire.

Artigue, M. (1999). What can we learn from educational research at the university level? Crucial questions for contemporary research in education. *Notices of the American Mathematical Society, 46*, 1377-1385.

Artigue, M. (2001). What can we learn from educational research at the university level? In D. Holton (Ed.), *The teaching and learning of mathematics at university level: An ICMI study* (pp. 207-220). Dordrecht, The Netherlands: Kluwer Academic Publishers.

Baker, B., Cooley, L., & Trigueros, M. (2000). A graphing calculus schema. *Journal for Research in Mathematics Education*, 31, 557-578.

Ball, D. (1989). *Teaching mathematics for understanding: What do teachers need to know about the subject matter?* East Lansing, MI: National Center for Research in Mathematics Education.

Bookman, J. (2001). Program evaluation and undergraduate mathematics renewal: The impact of calculus reform on student performance in subsequent courses. In S. Ganter (Ed.), *Calculus Renewal: Issues for Undergraduate Mathematics Education in the Next Decade.* New York: Plenum Press.

Bookman, J., & Friedman, C. P. (1994). A comparison of the problem solving performance of students in lab based and traditional calculus. In E. Dubinsky, A. H. Schoenfeld, & J. Kaput, *Research in collegiate mathematics education. I* (*CBMS Issues in Mathematics Education*, Vol. 4, pp. 101-116). Providence, RI: American Mathematical Society.

Brousseau, G. (1997). *Theory of didactical situations in mathematics*. (N. Balacheff, M. Cooper, R. Southland, & V. Warfield, Eds. and Trans.) Dordrecht, The Netherlands: Kluwer Academic Publishers.

Burmeister, S. L., Kenney P. A., & Nice, D. L. (1996). Analysis of effectiveness of supplement instruction (SI) sessions for college algebra, calculus, and statistics. In J. Kaput, A. H. Schoenfeld, & E. Dubinsky (Eds), *Research in collegiate mathematics education. II* (*CBMS Issues in Mathematics Education*, Vol. 6, pp. 145-154). Providence, RI: American Mathematical Society.

Carlson, M. P. (1998). A cross-sectional investigation of the development of the function concept. In A. H. Schoenfeld, J. Kaput and E. Dubinsky (Eds.), *Research in collegiate mathematics education*. III (*CBMS Issues in Mathematics Education*, Vol. 7, pp. 114-162). Providence, Rhode Island: American Mathematical Society.

Carnine, D., & Gersten, R. (2000). The nature and roles of research in improving achievement in mathematics. *Journal for Research in Mathematics Education*, *31*, 138-143.

Dahlberg, R. P., & Housman, D. L. (1997). Facilitating learning events through example generation. *Educational Studies in Mathematics*, 33, 283-299.

Dehaene, S. (1999). *The number sense: How the mind creates mathematics* (paperback ed.). New York: Oxford University Press.

Dubinsky, E., & McDonald, M. (2001). APOS: A constructivist theory of learning in undergraduate mathematics education. In D. Holton (Ed.), *The teaching and learning of mathematics at university level: An ICMI study* (pp. 275-282). Dordrecht, The Netherlands: Kluwer Academic Publishers.

Eisenhart, M., Borko, H., Underhill, R., Brown, C., Jones, D., & Agard, P. (1993). Conceptual knowledge falls through the cracks: Complexities of learning to teach mathematics for understanding. *Journal for Research in Mathematics Education*, 24, 8-40.

Fennema, E., & Franke, M. L. (1992). Teachers' knowledge and its impact. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning: A project of the National Council of Teachers of Mathematics* (pp. 147-164). New York, Macmillan.

Gibson, D. (1998). Students' use of diagrams to develop proofs in an introductory analysis course. In A. H. Schoenfeld, J. Kaput and E. Dubinsky (Eds.), *Research in collegiate mathematics education*. III (*CBMS Issues in Mathematics Education*, Vol. 7, pp. 284-307). Providence, Rhode Island: American Mathematical Society.

Hanna, G. (1998). Evaluating research papers in mathematics education. In A. Sierpinska and J. Kilpatrick (Eds.), *Mathematics Education as a Research Domain: A Search for Identity*, Vol. 2 (pp. 399-407). An ICMI Study. Dordrecht: Kluwer Academic Publishers.

Harel, G., & Dubinsky, E. (Eds.) (1992). *The concept of function: Aspects of epistemology and pedagogy* (MAA Notes, Vol. 25). Washington, DC: Mathematical Association of America.

Hiebert, J. (1999). Relationships between research and the NCTM Standards. *Journal for Research in Mathematics Education*, *30*(1), 3-19.

Howe, R. (1999). Book review: Knowing and teaching elementary mathematics. *Notices of the American Mathematical Society*, *46*, 881-887.

Hoyles, C., Noss, R., & Pozzi, S. (2001). Proportional Reasoning in Nursing Practice. *Journal for Research in Mathematics Education*, *32*, 4-27.

Kieran, C. (1992). The learning and teaching of school algebra. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning: A project of the National Council of Teachers of Mathematics* (pp. 390-419). New York, Macmillan.

Kilpatrick, J. (1992). A history of research in mathematics education. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning: A project of the National Council of Teachers of Mathematics* (pp. 3-38). New York, Macmillan.

Kilpatrick, J., Swafford, & Findell, B. (Eds.). (2001). *Adding it up: Helping children learn mathematics*. Washington, DC: National Academy Press.

Klass, K-D., Zompro, O., Kristensen, N. P., & Adis, J. (2002). Mantophasmatodea: A new insect order with extant members in the afrotropics. *Science*, 296, 1456-1459.

Lampert, M. (1990). When the problem is not the question and the solution is not the answer: Mathematical knowing and teaching. *American Educational Research Journal, 27,* 29-63.

Lave, J., & Wenger, E. (1991). *Situated Learning and : Legitimate Peripheral Participation.* Cambridge, UK: Cambridge University Press.

Leinhardt, G. O., Zaslavsky, O., & Stein, M. (1990). Functions, graphs, and graphing: Tasks, learning and teaching. *Review of Education Research*, *60*, 1-64.

Lester, F. K., Jr., & Lambdin, D. V. (1998). The ship of Theseus and other metaphors for thinking about what we value in mathematics education research. In A. Sierpinska and J. Kilpatrick (Eds.), *Mathematics Education as a Research Domain: A Search for Identity*, Vol. 2 (pp. 415-425). An ICMI Study. Dordrecht: Kluwer Academic Publishers.

Lester, F. K., Jr., & Wiliam, D. (2000). The evidential basis for knowledge claims in mathematics education research. *Journal for Research in Mathematics Education*, *31*, 132-137.

Linn, M., & Kessel, C. (1996). Success in mathematics: Increasing talent and gender diversity among college majors. In J. Kaput, A. H. Schoenfeld, & E. Dubinsky (Eds.), *Research in collegiate mathematics education. II* (*CBMS Issues in Mathematics Education*, Vol. 6, pp. 101-144). Providence, RI: American Mathematical Society.

Mackenzie, D. (2002). NSF moves with VIGRE to force changes in academia. *Science*, 296, 1380-1390.

Mason, J. (2002). *Researching your own practice: The discipline of noticing.* London: Routledge Falmer.

McKnight, C., Magid, A., Murphy, T. J., & McKnight, M. (2000). *Mathematics education research: A guide for the research mathematician*. Providence, RI: American Mathematical Society.

Miller, G. A. (1956). The magical number seven, plus or minus two: Some limits on our capacity for processing information. *Psychological Review*, 63, 81-97.

Mura, R. (1993). Images of mathematics held by university teachers of the mathematical sciences. *Educational Studies in Mathematics*, *25*, 375-385.

Mura, R. (1995). Images of mathematics held by university teachers of mathematics education. *Educational Studies in Mathematics, 28*, 385-399.

Niss, M. (1999). Aspects of the nature and state of research in mathematics education. *Educational Studies in Mathematics*, *40*, 1-24.

Noss, R., &Hoyles, C. (1996). The visibility of meanings: Modeling the mathematics of banking. *International Journal of Computers for Mathematical Learning, 1*, 3-31.

Pirie, S. (1998). Working towards a design for qualitative research. In A. R. Teppo (Ed.), *Qualitative research methods in mathematics education* (*JRME Monograph* No. 9, pp. 79-97). Reston, VA: NCTM.

Reys, R. E. (2001). Doctorates in mathematics education -- an acute shortage. *Notices of the American Mathematical Society, 47,* 1267-1270.

Reys, R. E., Glasgow, B., Ragan, G. A., & Simms, K. W. (2001). Doctoral programs in mathematics education in the United States: A status report. In R. E. Reys & J. Kilpatrick, (Eds.), *One field, many paths: U.S. doctoral programs in mathematics education* (*CBMS Issues in Mathematics Education*, Vol. 9. pp. 19-40). Providence, RI: American Mathematical Society.

Roddick, C. (in press). Calculus reform and traditional students' use of calculus in an engineering mechanics course. In A. Selden, E. Dubinsky, G. Harel, & F. Hitt (Eds.), *Research in collegiate mathematics education. V* (*CBMS Issues in Mathematics Education*). Providence, RI: American Mathematical Society.

Roth, W.-M. (1999). Professionals read graphs (imperfectly?). In F. Hitt & M. Santos (Eds.), *Proceedings of the twenty first annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 385-391). Columbus, OH: Clearinghouse for Science, Mathematics, and Environmental Education.

Rowland, T. (2002). Generic proofs in number theory. In S. R. Campbell & R. Zazkis (Eds.), *Learning and teaching number theory: Research in cognition and instruction* (pp. 157-183). Westport, CT: Ablex Publishing.

Schmidt, W. H., McKnight, C. C., & Raizen, S. A. (1996). *Splintered vision: An investigation of U.S. science and mathematics education*. Boston, MA: Kluwer Academic Publishers.

Schoenfeld, A. H. (1994). Some notes on the enterprise (research in collegiate mathematics education, that is). In E. Dubinsky, A. H. Schoenfeld, & J. Kaput, *Research in collegiate mathematics education. I* (*CBMS Issues in Mathematics Education*, Vol. 4, pp. 1-19). Providence, RI: American Mathematical Society.

Schoenfeld, A. H. (1999). The core, the canon, and the development of research skills: Issues in the preparation of education researchers. In. E. Lagemann and L. Shulman (Eds.), *Issues in education research: Problems and possibilities* (pp. 166-202). New York: Jossey-Bass.

Schoenfeld, A. H. (2000). Purposes and methods of research in mathematics education. *Notices of the American Mathematical Society*, *47*, 641-649.

Schoenfeld, A. H. (2001). Purposes and methods of research in mathematics education. In D. Holton (Ed.), *The teaching and learning of mathematics at university level: An ICMI study* (pp. 221-236). Dordrecht, The Netherlands: Kluwer Academic Publishers.

Schwingendorf, K.E., McCabe G. P., & Kuhn, J. (2000). In E. Dubinsky, A. H. Schoenfeld, & J. Kaput (Eds.), *Research in collegiate mathematics education. I*V (*CBMS Issues in Mathematics Education*, Vol. 8, pp. 63-76). Providence, RI: American Mathematical Society.

Selden, A., & Selden, J. (1993). Collegiate mathematics education research: What would that be like? *The College Mathematics Journal*, 24, 431-445.

Selden, A., & Selden, J. (2001). Tertiary mathematics education research and its future. In D. Holton (Ed.), *The teaching and learning of mathematics at university level: An ICMI study* (pp. 237-254). Dordrecht, The Netherlands: Kluwer Academic Publishers.

Selden, A., & Selden, J. (2001, September). Of what does mathematical knowledge consists? *Research Sampler*, *1*. MAA Online. Retrieved June 16, 2002, from http://www.maa.org/t_and_l/sampler/rs_1.html.

Selden, A. & Selden, J. (in press). Validations of proofs considered as texts: Can undergraduates tell whether an argument proves a theorem?" *Journal for Research in Mathematics Education*.

Selden, A., Selden, J., Hauk, S., & Mason, A. (2000). Why can't calculus students access their knowledge to solve non-routine problems? In E. Dubinsky, A. H. Schoenfeld, & J. Kaput (Eds.), *Research in collegiate mathematics education. I*V (*CBMS

Issues in Mathematics Education, Vol. 8, pp. 63-76). Providence, RI: American Mathematical Society.

Selden, J., Mason, A., & Selden, A. (1989). Can average calculus students solve nonroutine problems? *The Journal of Mathematical Behavior*, 8, 45-50.

Selden, J., Selden, A., & Mason, A. (1994). Even good calculus students can't solve nonroutine problems. In J. Kaput and E. Dubinsky (Eds.), *Research Issues in Undergraduate Mathematics Learning: Preliminary Analyses and Results* (MAA Notes No. 33, pp. 19-26). Washington, DC: Mathematical Association of America.

Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics*, *22*, 1-36.

Shure, P. (2001). The scholarship of learning and teaching: A look backward and a look ahead. AWM-MAA Invited Address. MathFest, University of Wisconsin at Madison. Unpublished.

Sierpinska, A. (1992). On understanding the notion of function. In G. Harel and E. Dubinsky (Eds.), *The concept of function: Aspects of epistemology and pedagogy* (MAA Notes Vol. 25, pp. 25-58). Washington, DC: Mathematical Association of America.

Smith, J. P., III. (1999). Tracking the mathematics of automobile production: Are schools failing to prepare students for work? *American Educational Research Journal*, *36*, 835-878.

Smith, E., Haarer, S., & Confrey, J. (1997). Seeking diversity in mathematics education: Mathematical modeling in the practice of biologists and mathematicians. *Science and Education*, (6)5, 441-472.

Szydlik, J. E. (2000). Mathematical beliefs and conceptual understanding of the limit of a function. *Journal for Research in Mathematics Education, 31*, 158-276.

Tall, D. (Ed.) (1991). *Advanced mathematical thinking*. Dordrecht: The Netherlands: Kluwer Academic Publishers.

Tall, D. and Vinner, S. (1981). Concept image and concept definition with particular reference to limits and continuity. *Educational Studies in Mathematics, 12*, 151-169.

Thompson, P.W. (1994). Students, functions, and the undergraduate curriculum. In E. Dubinsky, A. H. Schoenfeld, & J. Kaput, *Research in collegiate mathematics education. I* (*CBMS Issues in Mathematics Education*, Vol. 4, pp. 21-44). Providence, RI: American Mathematical Society.

Thorndike, E. L. (1922). *The psychology of arithmetic*. New York: Macmillan.

Trigueros, M., & Ursini, S. (in press). Starting college students' difficulties in working with different uses of variable. In A. Selden, E. Dubinsky, G. Harel, & F. Hitt (Eds.), *Research in collegiate mathematics education. V* (*CBMS Issues in Mathematics Education*). Providence, RI: American Mathematical Society.

Tsamir, P. (in press). Primary intuitions and instruction: The case of actual infinity. In A. Selden, E. Dubinsky, G. Harel, & F. Hitt (Eds.), *Research in collegiate mathematics education. V* (*CBMS Issues in Mathematics Education*). Providence, RI: American Mathematical Society.

White, P., & Mitchelmore, M. (1996) Conceptual knowledge in introductory calculus. *Journal for Research in Mathematics Education*, 27, 79-95.

Williams, S. R. (1991). Models of limit held by college calculus students. *Journal for Research in Mathematics Education*, *22*, 219-236.

Yackel, E., Rasmussen, & King, K. (2000) Social and sociomathematical norms in an advanced undergraduate mathematics class. *The Journal of Mathematical Behavior*, *19*, 275-287.

Zazkis, R., & Gunn. C. (1997). Sets, subsets, and the empty set: Students' constructions of mathematical conventions. *Journal of Computers in Mathematics and Science Teaching, 16*, 131-169.

August 26, 2002