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Abstract. This paper provides a partial generalization to matroid theory of the result of Halin and Jung that each simple graph with minimum vertex degree at least 4 has K_5 or the octahedron $K_{2,2,2}$ as a minor.

1. Introduction

The matroid notation and terminology used here will follow Oxley [4]. For a graph G, the associated simple graph will be denoted by $\widetilde{G}$. Similarly, the simple matroid associated with a matroid M will be denoted by $\widetilde{M}$. We shall use $\delta(G)$ to denote the minimum vertex degree of a graph G. The purpose of this paper is to present an extension of the following result of Halin and Jung [1].

Theorem 1.1. *If G is a simple graph such that $\delta(G) \geq 4$, then G has a K_5- or $K_{2,2,2}$-minor.*

It is natural when attempting to extend a graph result concerning vertex degrees to matroid theory to allow cocircuit size to play the role of vertex degree in graph theory. We denote the minimum cocircuit size of a matroid M by $g^*(M)$.

Theorem 1.2. *If M is a 3-connected binary matroid such that $g^*(M) \geq 4$, then M has a minor isomorphic to $M(K_{2,2,2})$, $M(K_5)$, $M^*(K_{3,3})$, or F_7.*

2. The Proof

The proof of Theorem 1.2 will use the following lemmas. The first is due to Hall [2].

Lemma 2.1. *If G is a 3-connected graph, then G has no $K_{3,3}$-minor if and only if either G is planar or $\widetilde{G} \cong K_5$.*

The remaining three lemmas are results of Seymour [5] and they are restated as Proposition 11.2.3, Lemma 11.2.8, and Theorem 13.2.2 in [4].

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$$A_{10} = \left[\begin{array}{c|ccccc} & 1 & 1 & 0 & 0 & 1 \\ & 1 & 1 & 1 & 0 & 0 \\ I_5 & 0 & 1 & 1 & 1 & 0 \\ & 0 & 0 & 1 & 1 & 1 \\ & 1 & 0 & 0 & 1 & 1 \end{array}\right] \qquad A_{12} = \left[\begin{array}{c|cccccc} & 1 & 1 & 1 & 0 & 0 & 0 \\ & 1 & 1 & 0 & 1 & 0 & 0 \\ I_6 & 1 & 0 & 0 & 0 & 1 & 0 \\ & 0 & 1 & 0 & 0 & 0 & 1 \\ & 0 & 0 & 1 & 0 & 1 & 1 \\ & 0 & 0 & 0 & 1 & 1 & 1 \end{array}\right]$$

FIGURE 1. $GF(2)$ representations of R_{10} and R_{12}.

Lemma 2.2. *If M is a 3-connected binary matroid, then M has no F_7^*-minor if and only if either M is regular or $M \cong F_7$.*

The next lemmas involve the matroids R_{10} and R_{12}. The matrices A_{10} and A_{12} shown in Figure 1 are $GF(2)$-representations of R_{10} and R_{12}, respectively.

Lemma 2.3. *Let e be an element of R_{10}. Then $R_{10}/e \cong M^*(K_{3,3})$.*

Lemma 2.4. *Let M be a 3-connected regular matroid. Then either M is graphic or cographic, or M has a minor isomorphic to one of R_{10} and R_{12}.*

Next we present the proof of Theorem 1.2.

Proof. Let M be a 3-connected binary matroid such that $g^*(M) \geq 4$. Suppose $M = M^*(G)$ for some graph G and has no minor isomorphic to $M^*(K_{3,3})$. Then G has no minor isomorphic to $K_{3,3}$. It follows from Lemma 2.1 that either G is planar or $G \cong K_5$. Thus M is either graphic or $M \cong M^*(K_5)$. If M is graphic then Theorem 1.1 implies that M has an $M(K_5)$-minor or an $M(K_{2,2,2})$-minor. On the other hand, if $M \cong M^*(K_5)$, then M has cocircuits of size 3; a contradiction. We conclude that the result holds if M is cographic.

Now suppose M is a 3-connected regular matroid and $g^*(M) \geq 4$. Then Lemma 2.4 implies that M is either graphic or cographic, or has a minor isomorphic to R_{10} or R_{12}. Since the result holds if M is graphic or cographic, we may assume that M has a minor isomorphic to R_{10} or R_{12}. If M has an R_{10}-minor then it follows from Lemma 2.3 that M has an $M^*(K_{3,3})$-minor. We may now assume that M has an R_{12}-minor. As the matroid R_{12} is regular but not graphic, it follows that R_{12} has a minor isomorphic to $M^*(K_5)$ or $M^*(K_{3,3})$. Since R_{12} is self-dual, we conclude that it has an $M(K_5)$- or $M^*(K_{3,3})$-minor. Thus M has such a minor.

Now suppose M is a 3-connected non-regular binary matroid so that $g^*(M) \geq 4$. Then M has an F_7- or F_7^*-minor. If M has an F_7-minor then the result holds, so we may assume that M has an F_7^*-minor. It follows from Lemma 2.2 that $M \cong F_7^*$. However F_7^* has cocircuits of size 3; a contradiction. We conclude that the result holds for all 3-connected binary matroids. □

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