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COMPUTATION OF NON-COMMUTATIVE GRÖBNER BASES IN GRASSMANN AND CLIFFORD ALGEBRAS

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Computation of Non-Commutative Gröbner Bases in Grassmann and Clifford Algebras

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Abstract. Tensor, Clifford and Grassmann algebras belong to a wide class of non-commutative algebras that have a Poincaré-Birkhoff-Witt (PBW) "monomial" basis. The necessary and sufficient condition for an algebra to have the PBW basis has been established by T. Mora and then V. Levandovskyy as the so called "non-degeneracy condition". This has led V. Levandovskyy to a re-discovery of the so called G-algebras (previously introduced by J. Apel) and GR-algebras (Gröbner-ready algebras). It was T. Mora who already in the 1990s considered a comprehensive and algorithmic approach to Gröbner bases for commutative and non-commutative algebras. It was T. Stokes who eighteen years ago introduced Gröbner left bases (GLB) and Gröbner left ideal bases (GLIB) for left ideals in Grassmann algebras, with the GLIB bases solving an ideal membership problem. Thus, a natural question is to first seek Gröbner bases with respect to a suitable admissible monomial order for ideals in tensor algebras T and then consider quotient algebras T/I. It was shown by Levandovskyy that these quotient algebras possess a PBW basis if and only if the ideal I has a Gröbner basis. Of course, these quotient algebras are of great interest because, in particular, Grassmann and Clifford algebras of a quadratic form arise this way. Examples of G-algebras include the quantum plane, universal enveloping algebras of finite dimensional Lie algebras, some Ore extensions, Weyl algebras and their quantizations, etc. Examples of GR-algebras, which are either G algebras or are isomorphic to quotient algebras of a G-algebra modulo a proper two-sided ideal, include Grassmann and Clifford algebras. After recalling basic concepts behind the theory of commutative Gröbner bases, a review of the Gröbner bases in PBW algebras, G-, and GR-algebras will be given with a special emphasis on computation of such bases in Grassmann and Clifford algebras. GLB and GLIB bases will also be computed.

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1. Introduction

The theory of Gröbner bases in multivariate polynomial rings over a field is very well known, see for example [9, 14, 15, 18, 19, 21, 43] and references therein. A multitude of various applications of these bases has been described in literature including but not limited to [16, 19, 27], and a great online resource of new applications can be found at [45]. An algorithmic approach to the theory of Gröbner bases and their applications in algebraic geometry has been successfully advocated in [19] and, in general, to the theory of commutative algebras in [28].

The concept of Gröbner bases in non-commutative algebras is much more recent and is not as well known. One of the earliest known, at least to this author, attempts to extend Gröbner bases theory to the non-commutative setting can be found in [8] and [43] whereas Stokes in [51] defines and computes various Gröbner bases in Grassmann algebras. Algorithmic methods relying on computing Gröbner bases in non-commutative left Poincaré-Birkhoff-Witt (PBW) rings, which are algebras and left noetherian domains at the same time, are studied in [17]. There, one finds algorithms to compute Gröbner bases in PBW rings which include quantum groups, Weyl algebras, enveloping algebras of finite dimensional Lie algebras, and Ore extensions, to name a few.

Grassmann and Clifford algebras are not, in general, PBW rings as they are not, in general, domains. Furthermore, as shown by Levandovskyy [35–37], they are obtainable as quotients of the so called G-algebras which possess a PBW basis consisting of generalized monomials, modulo a suitable ideal. Such quotient algebras Levandovskyy calls GR-algebras and they also include, for example, algebras given by structure constants [20]. Since in this paper we are interested in understanding mathematical and algorithmic differences between the commutative and the non-commutative settings, and especially in the case of Grassmann and Clifford algebras, we point out that G- and GR-algebras are the basic computational objects in Plural [38, 39, 50]. Furthermore, we will show computations of Gröbner bases in Grassmann algebras with a new Maple package TNB [11] which is based on Stokes' algorithms for computing the GLB and GLIB bases [51].

The organization of this paper is as follows. In Section 2 we provide a brief review of Gröbner bases in polynomial rings. In Section 3 we give examples of using commutative Gröbner bases: in symbolic computations in Clifford algebras, in robotics, and when working with symmetric polynomials. In Section 4, following [17], we define PBW rings and provide their basic properties. In Section 5, following [39], we define and show basic properties of G- and GR-algebras. In Section 6 we show computations of Gröbner bases in Grassmann and Clifford algebras with Plural and TNB. In Section 7 we discuss GLB and GLIB bases introduced by Stokes [51] for Grassmann algebras In Section 8 we summarize computational differences and similarities when computing Gröbner bases in polynomial rings and in Grassmann and Clifford algebras.

2. Gröbner bases in polynomial rings

Our main reference is [19]. In particular, $k[x_1, \ldots, x_n]$ is a polynomial ring in indeterminates $x_1, \ldots, x_n$ over a field k whereas $\mathbf{V}(f_1, \ldots, f_s)$ is an **affine variety** viewed as a subset of k^n consisting of common zeros of polynomials $f_1, \ldots, f_s \in k[x_1, \ldots, x_n]$. In particular, $\langle f_1, \ldots, f_s \rangle$ denotes an ideal in $k[x_1, \ldots, x_n]$ generated by the polynomials. We say that ideal $I \subset k[x_1, \ldots, x_n]$ is **finitely generated** if there exist $f_1, \ldots, f_s \in k[x_1, \ldots, x_n]$ such that $I = \langle f_1, \ldots, f_s \rangle$. Then we say that $f_1, \ldots, f_s$ are a **basis** of I.

Proposition 2.1 (Cox). *If $f_1, \ldots, f_s$ and $g_1, \ldots, g_t$ are bases of the same ideal in $k[x_1, \ldots, x_n]$, so that $\langle f_1, \ldots, f_s \rangle = \langle g_1, \ldots, g_t \rangle$, then $\mathbf{V}(f_1, \ldots, f_s) = \mathbf{V}(g_1, \ldots, g_t)$.*

Example 1. Let $f_1 = 4(x_1-1)^2 + 4x_2^2 + 4x_3^2 - 9$ and $f_2 = (x_1+1)^2 + x_2^2 + x_3^2 - 4$ be in $\mathbb{R}[x_1, x_2, x_3]$. Then, $\mathbf{V}(f_1)$ and $\mathbf{V}(f_2)$ are two spheres viewed as varieties. The intersection of these varieties is a circle C. Since we have equality of ideals $I = \langle f_1, f_2 \rangle = \langle 256x_2^2 + 256x_3^2 - 495, 16x_1 - 7 \rangle$,

$$C = \mathbf{V}(f_1, f_2) = \mathbf{V}(f_1) \cap \mathbf{V}(f_2) = \mathbf{V}(256x_2^2 + 256x_3^2 - 495, 16x_1 - 7). \tag{2.1}$$

Thus, C can also be viewed as the intersection of a cylinder $256x_2^2 + 256x_3^2 = 495$ with a plane $16x_1 = 7$.

In fact, as we will see later, polynomials $g_1 = 256x_2^2 + 256x_3^2 - 495$ and $g_2 = 16x_1 - 7$ in the above example give a Gröbner basis for the ideal I.

In order to introduce Gröbner basis for polynomial ideals in $k[x_1, \ldots, x_n]$, one needs to define first a monomial ordering.

Definition 2.2. A **monomial ordering** on $k[x_1, \ldots, x_n]$ is any relation $>$ on $\mathbb{Z}^n_{\geq 0} = \{(\alpha_1, \ldots, \alpha_n) \,|\, \alpha_i \in \mathbb{Z}_{\geq 0}\}$, or equivalently, any relation on the set of monomials x^α, $\alpha \in \mathbb{Z}^n_{\geq 0}$, satisfying:

(i) $>$ is a total ordering on $\mathbb{Z}^n_{\geq 0}$ (for any $\alpha, \beta \in \mathbb{Z}^n_{\geq 0}$, $\alpha > \beta, \alpha = \beta$ or $\beta > \alpha$)
(ii) If $\alpha > \beta$ and $\gamma \in \mathbb{Z}^n_{\geq 0}$, then $\alpha + \gamma > \beta + \gamma$.
(iii) $>$ is a well-ordering on $\mathbb{Z}^n_{\geq 0}$ (every non-empty subset has smallest element)

A few standard monomial orders are as follows:

- **Lexicographic Order**: $\alpha >_{lex} \beta$ if, in the vector difference $\alpha - \beta \in \mathbb{Z}^n$, the left-most non-zero entry is positive.
- **Graded Lex Order**: $\alpha >_{grlex} \beta$ if either $|\alpha| > |\beta|$, or $|\alpha| = |\beta|$ and $\alpha >_{lex} \beta$. Here $|\alpha| = \sum_{i=1}^n \alpha_i$.
- **Graded Reverse Lex Order**: $\alpha >_{grevlex} \beta$ if either $|\alpha| > |\beta|$, or $|\alpha| = |\beta|$ and in $\alpha - \beta \in \mathbb{Z}^n$ the right-most non-zero entry is negative.
- **Graded Inverse Lex Order**: $\alpha >_{ginvlex} \beta$ if either $|\alpha| > |\beta|$, or $|\alpha| = |\beta|$ and in $\alpha - \beta \in \mathbb{Z}^n$ the right-most non-zero entry is positive.[1]

[1] The Graded Inverse Lex Order `Deg[InvLex]` and the Graded Lex Order `Deg[Lex]` will later be used as admissible orders `drp` and `dp` when computing with Plural [50] in Grassmann and Clifford algebras.

Once a monomial order $>$ has been chosen, one can then determine the leading term $\mathrm{LT}(f)$ in each polynomial f, and order any two monomials. This in turn allows one to introduce the division algorithm

Theorem 2.3 (General Division Algorithm). *Fix a monomial order $>$ on $\mathbb{Z}^n_{\geq 0}$, and let $F = (f_1, \dots, f_s)$ be an ordered s-tuple of polynomials. Then every $f \in k[x_1, \dots, x_n]$ can be written as*

$$f = a_1 f_1 + \cdots a_s f_s + r, \tag{2.2}$$

where $a_i, r \in k[x_1, \dots, x_n]$ and either $r = 0$ or r is a linear combination, with coefficients in k, of monomials, none of which is divisible by any of $\mathrm{LT}(f_1), \dots, \mathrm{LT}(f_s)$*. We call r a* **remainder** *of f on division by F. Furthermore, if $a_i f_i \neq 0$, then we have* $\mathrm{multideg}(f) \geq \mathrm{multideg}(a_i f_i)$.

Remark 2.4. The remainder r in (2.2) (and the quotient monomials a_i), is not unique as it depends on the monomial order and on the division order of f by the polynomials in F. This last shortcoming of the Division Algorithm disappears when we divide polynomials by a Gröbner basis.

Remark 2.5. The termination of the Division Algorithm in $k[x_1, \dots, x_n]$ is guaranteed by the fact that $k[x_1, \dots, x_n]$ is a noetherian ring. For the actual algorithm, see for example [19] or [21].

The next concept needed is that of a monomial ideal so that we can state Dickson's Lemma and, as the main foundation for the theory, Hilbert Basis Theorem.

Definition 2.6. An ideal $I \subset k[x_1, \dots, x_n]$ is a **monomial ideal** if there is a subset $A \subset \mathbb{Z}^n_{\geq 0}$ (possibly infinite) such that I consists of all polynomials which are finite sums of the form $\sum_{\alpha \in A} h_\alpha x^\alpha$, where $h_\alpha \in k[x_1, \dots, x_n]$. In this case we write $I = \langle x^\alpha \,:\, \alpha \in A \rangle$.

Lemma 2.7 (Dickson). *Every monomial ideal $I \subset k[x_1, \dots, x_n]$ has a finite basis.*

For the proof of Dickson's Lemma see for example [19] where it is used to prove the following critical result.

Proposition 2.8. *Let $I \subset k[x_1, \dots, x_n]$ be an ideal.*

(i) $\langle \mathrm{LT}(I) \rangle$ *is a monomial ideal.*
(ii) *There are finitely-many $g_1, \dots, g_s \in I$ such that*

$$\langle \mathrm{LT}(I) \rangle = \langle \mathrm{LT}(g_1), \dots, \mathrm{LT}(g_s) \rangle.$$

In the above, $\mathrm{LT}(I)$ denotes the set of leading terms of elements of a non-zero ideal $I \subset k[x_1, \dots, x_n]$ and $\langle \mathrm{LT}(I) \rangle$ is the ideal generated by the elements of $\mathrm{LT}(I)$. Dickson's Lemma is used in [19] to prove the famous Hilbert Basis Theorem:

Theorem 2.9 (Hilbert Basis Theorem). *Every ideal $I \subset k[x_1, \dots, x_n]$ has a finite generating set. That is, $I = \langle g_1, \dots, g_s \rangle$ for some $g_1, \dots, g_s \in I$.*

Now that we know that every ideal in $k[x_1, \ldots, x_n]$ is finitely generated, we are ready to define a Gröbner basis for an ideal $I \subset k[x_1, \ldots, x_n]$.

Definition 2.10. Fix a monomial order. A finite subset $G = \{g_1, \ldots, g_t\}$ of an ideal I is said to be a **Gröbner basis** for I if

$$\langle \mathrm{LT}(g_1), \ldots, \mathrm{LT}(g_t) \rangle = \langle \mathrm{LT}(I) \rangle. \tag{2.3}$$

As a consequence of Proposition 2.8 and the Hilbert Basis Theorem we have

Corollary 2.11. *Fix a monomial order. Then every ideal $I \subset k[x_1, \ldots, x_n]$ other than $\{0\}$ has a Gröbner basis.*

Before we describe how one can compute Gröbner bases introduced by Bruno Buchberger [14, 15], let us summarize some useful facts [19]:

Proposition 2.12. *If $g_1, \ldots, g_t$ is a Gröbner basis for I and $f \in k[x_1, \ldots, x_n]$, then $f \in I$ if and only if the remainder of f on division by $g_1, \ldots, g_t$ is zero.*

The above proposition provides a way of solving the so called *Ideal Membership Problem* as it allows us to decide whether a given polynomial $f \in k[x_1, \ldots, x_n]$ belongs to an ideal I : it is enough to compute a Gröbner basis G for I in any monomial order and divide f by G. Denote the remainder of this division as $r = \overline{f}^G$. By the proposition, $f \in I$ if and only if $r = 0$. In general, due to the uniqueness of r, one gets unique coset representatives for elements in the quotient ring $k[x_1, \ldots, x_n]/I$: The coset representative of $[f] \in k[x_1, \ldots, x_n]/I$ will be $\overline{f}^G$.

Proposition 2.13. *If $g_1, \ldots, g_t$ is a Gröbner basis for I and $f \in k[x_1, \ldots, x_n]$, then f can be written uniquely in the form $f = g + r$ where $g \in I$ and no term of r is divisible by any* $\mathrm{LT}(g_i)$.

This last proposition gives a practical criterion that describes when the division of f by a Gröbner basis $G = \{g_1, \ldots, g_t\}$ stops: when no term of the remainder is divisible by any $\mathrm{LT}(g_i)$.

An answer for how to compute Gröbner basis is provided by Buchberger's algorithm and criterion (and its modifications) that use S-polynomials. This approach through S-polynomials is later generalized to non-commutative rings.

Definition 2.14. The S-**polynomial** of $f_1, f_2 \in k[x_1, \ldots, x_n]$ is defined as

$$S(f_1, f_2) = \frac{x^\gamma}{\mathrm{LT}(f_1)} f_1 - \frac{x^\gamma}{\mathrm{LT}(f_2)} f_2, \tag{2.4}$$

where $x^\gamma = \mathrm{lcm}(\mathrm{LM}(f_1), \mathrm{LM}(f_2))$ and $\mathrm{LM}(f_i)$ is the leading monomial of f_i w.r.t. some monomial order.

Theorem 2.15 (Buchberger Theorem). *A basis $\{g_1, \ldots, g_t\} \subset I$ is a Gröbner basis of I if an only if $\overline{S(g_i, g_j)}^G = 0$ for all $i < j$.*

Example 2. Let $f_1 = x^4 - 3xy, f_2 = x^2y - 2 \in k[x, y]$ and *lex* order with $x > y$. Then, $\mathrm{LT}(f_1) = x^4$, $\mathrm{LT}(f_2) = x^2y$ and

$$S(f_1, f_2) = \frac{x^4y}{x^4} \cdot f_1 - \frac{x^4y}{x^2y} \cdot f_2 = y \cdot f_1 - x^2 \cdot f_2 = -3xy^2 + 2x^2 \in \langle f_1, f_2 \rangle.$$

Since $\mathrm{LT}(S(f_1, f_2))$ divisible by neither $\mathrm{LT}(f_1)$ nor $\mathrm{LT}(f_2)$, or, $\mathrm{LT}(S(f_1, f_2)) \notin \langle \mathrm{LT}(f_1), \mathrm{LT}(f_2) \rangle$, we see that f_1, f_2 is *not* a Gröbner basis of $\langle f_1, f_2 \rangle$.

Buchberger's algorithm for finding a Gröbner basis can be described as follows:

Buchberger's Algorithm. *Given* $\{f_1, \dots, f_s\} \subset k[x_1, \dots, x_n]$, *consider the algorithm which starts with* $F = \{f_1, \dots, f_s\}$ *and then repeats the two steps*

- *(Compute Step) Compute* $\overline{S(f_i, f_j)}^F$ *for all* $f_i, f_j \in F$ *with* $i < j$,
- *(Augment step) Augment* F *by adding the non-zero* $\overline{S(f_i, f_j)}^F$ *until the Compute Step gives only zero remainders. The algorithm always terminates and the final value of* F *is a Gröbner basis of* $\langle f_1, \dots, f_s \rangle$.

We will see later that all of the above steps from defining a monomial order through defining a Gröbner basis, S-polynomials, and a new algorithm in the non-commutative cases of interest to us – Grassmann and Clifford algebras – will be in principle repeated with certain modifications that will need to account for non-commutativity of these algebras and for the fact that, in general, these algebras unlike $k[x_1, \dots, x_n]$ are not domains.

Example 3. Let $F_1 = \{f_1, f_2\}$ where $f_1 = 4(x_1 - 1)^2 + 4x_2^2 + 4x_3^2 - 9$ and $f_2 = (x_1 + 1)^2 + x_2^2 + x_3^2 - 4$ are as in Example 1. For the monomial order *lex* order with $x_1 > x_2 > x_3$, we find $f_3 = \overline{S(f_1, f_2)}^{F_1} = -16x_1 + 7$, so we extend F_1 to $F_2 = \{f_1, f_2, f_3\}$. Then, $f_4 = \overline{S(f_1, f_3)}^{F_2} = 495 - 256x_2^2 - 256x_3^2$, so we extend F_2 to $F_3 = \{f_1, f_2, f_3, f_4\}$. Next we find that $\overline{S(f_1, f_4)}^{F_3} = 0$. Thus, we have

$$\overline{S(f_1, f_2)}^{F_3} = \overline{S(f_1, f_3)}^{F_3} = \overline{S(f_1, f_4)}^{F_3} = 0.$$

Furthermore, we find that $\overline{S(f_2, f_3)}^{F_3} = \overline{S(f_2, f_4)}^{F_3} = \overline{S(f_3, f_4)}^{F_3} = 0$. Since $\overline{S(f_i, f_j)}^{F_3} = 0$ for all $i < j$ and $f_i, f_j \in F_3$, we conclude that a Gröbner basis for $I = \langle f_1, f_2 \rangle$ finally is

$$\begin{aligned} F_3 = \{&4(x_1 - 1)^2 + 4x_2^2 + 4x_3^2 - 9,\ (x_1 + 1)^2 + x_2^2 + x_3^2 - 4, \\ &- 16x_1 + 7, 495 - 256x_2^2 - 256x_3^2\}. \end{aligned} \tag{2.5}$$

Before we show specific computational examples of applying Gröbner bases in polynomial rings, we need to make the following observations:

(i) Gröbner basis F_3 shown in (2.5) is too big: A standard way to reduce it is to replace any polynomial f_i with its remainder on division by
$$\{f_1, \dots, f_{i-1}, f_{i+1}, \dots, f_t\},$$
removing zero remainders, and for polynomials that are left, making their leading coefficient equal to 1. This produces a *reduced Gröbner basis*. In general, for a fixed monomial order, any ideal in $k[x_1, \dots, x_n]$ has a *unique* reduced Gröbner basis [19,21,28].

Example 4. We reduce Gröbner basis F_3 for the ideal I shown in (2.5) as follows: We compute
$$\overline{f_1}^{F_3\setminus\{f_1\}} = \overline{f_2}^{F_4\setminus\{f_2\}} = 0$$
where $F_4 = \{f_2, f_3, f_4\}$. So, we set $F_5 = \{f_3, f_4\}$ and find that
$$\overline{f_3}^{F_5\setminus\{f_3\}} = -16x_1 + 7 = f_3, \quad \overline{f_4}^{F_5\setminus\{f_4\}} = 495 - 256x_2^2 - 256x_3^2 = f_4,$$
so a reduced Gröbner basis for the ideal $I = \langle f_1, f_2 \rangle$ from Example 3 is
$$G_{red} = \{x_1 - \tfrac{7}{16}, x_2^2 + x_3^2 - \tfrac{495}{256}\}. \tag{2.6}$$
It is essentially the same basis as the one shown in (2.1).

(ii) The reduced Gröbner basis G is characterized by two features: (a) the leading coefficient of each polynomial p in G is 1, and (b) that for all p in G, no monomial of p lies in the monomial ideal $\langle \mathrm{LT}(G \setminus \{p\}) \rangle$. Observe, that in *lex* order with $x_1 > x_2 > x_3$, none of the terms of $x_1 - \frac{7}{16}$ belongs to $\langle \mathrm{LT}(x_2^2 + x_3^2 - \frac{495}{256}) \rangle = \langle x_2^2 \rangle$ and, likewise, none of the terms of $x_2^2 + x_3^2 - \frac{495}{256}$ belongs to $\langle \mathrm{LT}(x_1 - \frac{7}{16}) \rangle = \langle x_1 \rangle$.

(iii) Buchberger's Algorithm has been made more efficient, see [9,19,28] and references therein. See also [23] although `FGb` package is based on a different algorithm than Buchberger's.

3. Examples of applying commutative Gröbner bases

A list of problems that can be solved using Gröbner bases constantly grows. Here is a partial list of some problems that can be solved with their help:

- The ideal membership problem, i.e., does $f \in I = \langle f_1, \dots, f_s \rangle$? See [19,21].
- Finding generators for the intersection of two ideals $I \cap J$ [19].
- Solving systems of polynomial equations, e.g., intersecting surfaces and curves, finding closest point on curve to the given point, Lagrange multiplier problems (especially for several multipliers), etc. [18,19,27,28].
- Finding curves and surfaces equidistant, respectively, to a given curve and surface [7,19].
- The implicitization problem, i.e., eliminating parameters and finding implicit forms for curves and surfaces [18,19].
- The forward and the inverse kinematic problems in robotics [19].

- Automatic geometric theorem proving [15,19].
- Expressing invariants of a finite group, e.g., symmetric polynomials, in terms of generating invariants [19,52].
- Finding relations between polynomial functions, e.g., interpolating functions (syzygy relations)[2] [5,19].
- For many other applications, including Integer programming, complex information systems, or algebraic coding theory see [18] and references therein, and [16,19,27].
- See also Bibliography on Gröbner bases at RICAM [45].

We will show a few examples, some of which are related to Grassmann and Clifford algebras. Our first example shows how one can solve symbolic equations, constraints, or systems of constraints in order to find a general element in a Grassmann or Clifford algebra that satisfies the constraint(s).

Example 5 (Idempotent variety in $C\ell_{2,0}$). Consider Clifford algebra $C\ell_{2,0}$ with a Grassmann monomial basis $1, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_{12}$ where $\mathbf{e}_{12} = \mathbf{e}_1 \wedge \mathbf{e}_2$. What is the most general idempotent $u \in C\ell_{2,0}$? That is, we are looking for the most general element $u \in C\ell_{2,0}$ such that $u^2 = u$. Let $u = x_0 + x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_{12}\mathbf{e}_{12}$. Then, the equation $u^2 = u$ yields the following four polynomial equations:

$$\begin{aligned} p_1 &= x_0^2 + x_1^2 + x_2^2 - x_{12}^2 - x_0, & p_2 &= x_1(2x_0 - 1), \\ p_3 &= x_2(2x_0 - 1), & p_4 &= x_{12}(2x_0 - 1) \end{aligned} \tag{3.1}$$

Thus, the family of idempotents is an affine variety $\mathbf{V}(p_1, p_2, p_3, p_4)$. We will solve the above system by finding a reduced Gröbner basis G for the ideal $I\langle p_0, p_1, p_2, p_3\rangle$ in $\mathbb{R}[x_0, x_1, x_2, x_{12}]$ for *lex* order with $x_0 > x_1 > x_2 > x_{12}$. The basis G consists of seven polynomials:

$$\begin{aligned} g_1 &= x_{12}(4x_1^2 - 1 + 4x_2^2 - 4x_{12}^2), & g_2 &= x_2(4x_1^2 - 1 + 4x_2^2 - 4x_{12}^2), \\ g_3 &= x_1(4x_1^2 - 1 + 4x_2^2 - 4x_{12}^2), & g_4 &= x_{12}(2x_0 - 1), \\ g_5 &= x_2(2x_0 - 1), & g_6 &= x_1(2x_0 - 1), \\ g_7 &= x_0^2 + x_2^2 - x_{12}^2 + x_1^2 - x_0. \end{aligned} \tag{3.2}$$

Notice that $g_1, g_2, g_3 \in G_1 = G \cap \mathbb{R}[x_1, x_2, x_{12}]$ whereas $g_4, g_5, g_6, g_7 \in G_0 = G \subset \mathbb{R}[x_0, x_1, x_2, x_{12}]$. Thus, $\mathbf{V}(p_1, p_2, p_3, p_4) = \mathbf{V}(g_1, g_2, g_3, g_4, g_5, g_6, g_7)$. It is now easy to find first solution when $x_0 = \frac{1}{2}$:

$$u_{1,2} = \tfrac{1}{2} + x_{12}\mathbf{e}_{12} \pm \tfrac{1}{2}\sqrt{1 - 4x_2^2 + 4x_{12}^2}\,\mathbf{e}_1 + x_2\mathbf{e}_2 \tag{3.3}$$

provided $1 - 4x_2^2 + 4x_{12}^2 \geq 0$. Second solution when $x_0 \neq 0$ requires $x_1 = x_2 = x_{12} = 0$ and yields trivial idempotents 0 and ± 1. Thus, $u_{1,2}$ in (3.3) are the only non-trivial idempotents in $C\ell_{2,0}$ and their variety is the surface and the inside of

[2] Procedure `SyzygyIdeal` from SP package [5] finds syzygy relations, if any, between polynomials $F = (f_1, \ldots, f_m) \subset k[x_1, \ldots, x_n]$ by computing a Gröbner basis for the n-th elimination ideal I_F (the *ideal of relations*) of the ideal $J_F = \langle f_1 - y_1, f_2 - y_2, \ldots, f_m - y_m\rangle$ for the *lex* order with $x_1 > \ldots > x_n > y_1 > \ldots > y_m$. See [19].

the hyperboloid $4x_1^2 + 4x_2^2 - 4x_{12}^2 = 1$. The primitive idempotents $\frac{1}{2}(1 \pm \mathbf{e}_1)$ and $\frac{1}{2}(1 \pm \mathbf{e}_2)$ belong to this variety when $x_{12} = x_2 = 0$ and $x_{12} = x_1 = 0$, respectively. For a classification of families of general idempotents in Clifford algebras see [6].

Our second example is related to the screw theory represented in the language of Clifford algebra $C\ell_{0,3,1}$. This algebra contains a copy of the group of rigid motions $SE(3)$, its Lie algebra, the screws, and elements corresponding to points, lines and planes in Euclidean space $\mathbb{R}^3$. [48] In fact, in [49], Selig and Bayro-Corrochano take two copies of that algebra and use the Clifford algebra $C\ell_{0,6,2}$ to study momenta and inertia. In this example we show how symbolic computations in $C\ell_{0,6,2}$ can be performed modulo certain polynomial relations on elements of the rigid group.

Example 6 (Rigid Transformations). We follow notation from [49]. Let $C\ell_{0,6,2}$ be generated by $e_1, e_2, e_3, e, a_1, a_2, a_3, a$ such that $a_i^2 = e_1^2 = -1$ and $a^2 = e^2 = 0$. A general form of an element in $C\ell_{0,3,1} \subset C\ell_{0,6,2}$ to represent a rigid transformation, which is a combination of rotations and translations, is

$$g = \alpha_0 + \alpha_1 e_2 e_3 + \alpha_1 e_3 e_1 + \alpha_3 e_1 e_2 + \beta_0 e e_1 e_2 e_3 + \beta_1 e_1 e + \beta_2 e_2 e + \beta_3 e_3 e. \tag{3.4}$$

Here, g is subject to the condition $g^* g = 1$ where * denotes Clifford conjugation and $\alpha' s, \beta' s$ are real parameters. The condition expressed in terms of the parameters gives two polynomial identities:

$$\alpha_0^2 + \alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1 \quad \text{and} \quad \alpha_0\beta_0 + \alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3 = 0. \tag{3.5}$$

In $C\ell_{0,6,2}$, the authors represent co-screws (momenta) as the following elements:

$$\mathcal{P} = p_x a_2 a_3 + p_y a_3 a_1 + p_z a_1 a_2 + l_x a_1 a + l_y a_2 a + l_z a_3 a \tag{3.6}$$

which transform under the group of rigid motions as the ordinary screws with $a's$ replaced by $e's$. They define an involution $\bar{u}$ on $C\ell_{0,6,2}$ which exchanges $a's$ for $e's$ as follows: $\bar{a}_i = e_i, \bar{a} = e$, and $\bar{e}_i = a_i, \bar{e} = a$. Then, the group action on the moment is

$$\mathcal{P} \to \bar{g}\mathcal{P}\bar{g}^* \tag{3.7}$$

The evaluation map of a co-screw on a screw

$$\mathbf{s} = \omega_x e_2 e_3 + \omega_y e_3 e_1 + \omega_z e_1 e_2 + v_x e_1 e + v_y e_2 e + v_z e_3 e$$

they state as

$$\mathcal{P}(\mathbf{s}) = \mathcal{P} \vee (Q_0 \wedge \mathbf{s}) = (\mathcal{P} \wedge Q_0) \vee \mathbf{s} \tag{3.8}$$

where $\vee$ is the shuffle product from the Cayley-Grassmann algebra [48], [53] and Q_0 is the following element of $C\ell_{0,6,2}$:

$$Q_0 = a_2 a_3 e_1 e + a_3 a_1 e_2 e + a_1 a_2 e_3 e + a_1 a e_2 e_3 + a_2 a e_3 e_1 + a_3 a e_1 e_2. \tag{3.9}$$

It is not difficult to see that Q_0 is invariant under the exchange. It is not easy to see, as the authors claim, that Q_0 is invariant under the group action $(g\bar{g})Q_0(g\bar{g})^*$, that is, $Q_0 = (g\bar{g})Q_0(g\bar{g})^*$, yet Q_0 is not invariant under the action of g or $\bar{g}$ alone, $Q_0 \neq gQ_0g^*$ and $Q_0 \neq \bar{g}Q_0\bar{g}^*$. We can verify this claim using a Gröbner basis G for the ideal $\langle f_1, f_2 \rangle \subset \mathbb{R}[\alpha_0, \alpha_1, \alpha_2, \alpha_3, \beta_0, \beta_1, \beta_2, \beta_3]$ where f_1, f_2 are the polynomials defined by the relations (3.5) and then by reducing all coefficients of the product $(g\bar{g})Q_0(g\bar{g})^*$ modulo G. Since we are reducing modulo the Gröbner basis, remainders of the reduction are uniquely defined. The Gröbner basis G for the $lex(\alpha_0 > \alpha_1 > \alpha_2 > \alpha_3 > \beta_0 > \beta_1 > \beta_2 > \beta_3)$ contains four polynomials including the original two polynomials. Computing the difference we find

$$\begin{aligned}(g\bar{g})Q_0(g\bar{g})^* - Q_0 = {} & h_1 e_1 e_2 a_3 a + h_2 e_1 e_3 a_2 a + h_3 e_1 e a_1 a \\ & + h_4 e_1 e a_2 a_3 + h_5 e_2 e_3 a_1 a + h_6 e_2 e a_1 a_3 \\ & + h_7 e_2 e a_2 a + h_8 e_3 e a_1 a_2 + h_9 e_3 e a_3 a \end{aligned} \tag{3.10}$$

where $h_j \in \mathbb{R}[\alpha_0, \alpha_1, \alpha_2, \alpha_3, \beta_0, \beta_1, \beta_2, \beta_3]$ and $\overline{h_j}^G = 0$ for $j = 1, \ldots, 9$. Thus, indeed, Q_0 is invariant under the group action $(g\bar{g})Q_0(g\bar{g})^*$ where g is the rigid transformation. The same way one can show that Q_0 is not invariant under the action of g or $\bar{g}$ alone.

In general, the action on $\mathcal{P}$ shown in (3.7) needs to be computed modulo G as well. Later in their paper Selig and Bayro-Corrochano deduce that the inertia N must transform according to $N \to (g\bar{g})N(g\bar{g})^*$ and hand-compute such transformation of N when $g = 1 + \frac{1}{2}t_x e_1 e$. When g is more general, or as general as possible, hand computation is no longer practical and the above approach is superior.

Finally, we show a simple add-on procedure to CLIFFORD/Bigebra [3] that can reduce symbolic polynomial coefficients of any element in the defined Clifford algebra modulo a set of polynomial relations, e.g., as in (3.5). This approach is particularly useful when computing action of the Lipschitz group or the spin groups [40] modulo relations that coefficients of general elements of these groups must satisfy.

```
ReduceClipolynom:=proc(p::{cliscalar,clibasmon,climon,clipolynom})
                  local F,tmon,T,C,i,m,G:
F,tmon:=op(procname):
if type(p,clibasmon) then return p end if:
G:=Groebner:-Basis(F,tmon);
if type(p,cliscalar) then return Reduce(p,G,tmon) end if;
T:=convert(cliterms(p),list):
C:=[seq(coeff(p,m),m=T)];
C:=map(Groebner:-Reduce,C,G,tmon);
return add(C[i]*T[i],i=1..nops(T));
end proc:
```

The above procedure can be used as in `ReduceClipolynom[F,T]`(h_1) where $F = [r_1, \ldots, r_s]$ is a list of initial polynomial relations, T is the chosen monomial order, and h_1 is the argument polynomial. Notice that commands `Groebner:-Basis` and `Groebner:-Reduce` come from Maple's `Groebner` package. [41]

For our third example, we need the following result [19].[3]

Proposition 3.1. *Suppose that $f_1, \ldots, f_m \in k[x_1, \ldots, x_n]$ are given. Fix a monomial order $k[x_1, \ldots, x_n, y_1, \ldots, x_m]$ where any monomial involving one of $x_1, \ldots, x_n$ is greater than all monomials in $k[y_1, \ldots, y_m]$. Let G be a Gröbner basis of the ideal $J = \langle f_1 - y_1, \ldots, f_m - y_m \rangle \subset k[x_1, \ldots, x_n, y_1, \ldots, x_m]$. Given $f \in k[x_1, \ldots, x_n]$, let $g = \overline{f}^G$ be the remainder of f on division by G. Then*

(i) *$f \in k[f_1, \ldots, f_m]$ if and only if $g \in k[y_1, \ldots, y_m]$.*
(ii) *If $f \in k[f_1, \ldots, f_m]$, then $f = g(f_1, \ldots, f_m)$ is an expression of f as a polynomial in $f_1, \ldots, f_m$.*

Example 7 (Symmetric polynomials). Let G be the symmetric group S_3. Let

$$\sigma_1 = x_1 + x_2 + x_3, \quad \sigma_2 = x_1x_2 + x_1x_3 + x_2x_3, \quad \text{and} \quad \sigma_3 = x_1x_2x_3$$

be the *elementary symmetric polynomials* in x_1, x_2, x_3. [52] A Gröbner basis F for the ideal $I = \langle \sigma_1 - y_1, \sigma_2 - y_2, \sigma_3 - y_3 \rangle$ in $lex(x_1, x_2, x_3, y_1, y_2, y_3)$ order is

$$F = [x_3^3 - x_3^2y_1 + y_2x_3 - y_3,\ x_2^2 + x_2x_3 - x_2y_1 + x_3^2 - x_3y_1 + y_2,\ x_1 + x_2 + x_3 - y_1]$$

Let

$$f = x_1^2x_2 + x_1x_2^2 + 3x_1x_2x_3 + x_1^2x_3 + x_1x_3^2 + x_2^2x_3 + x_2x_3^2 - x_1^2x_2^2x_3^2.$$

It can be checked directly that $f(\mathbf{x}) = f(\sigma\mathbf{x}),\ \forall\sigma \in S_3$. That is, f is invariant under S_3 and $f \in k[x_1, x_2, x_3]^{S_3}$. Reducing f modulo F gives $g = \overline{f}^F = y_1y_2 - y_3^2 \in k[y_1, y_2, y_3]$. Thus, by part (i) of the above Proposition, we see again that f is symmetric. Furthermore, from part (ii) we get that $f = \sigma_1\sigma_2 - \sigma_3^2$.

For more examples on finite group generators and finding the so called *syzygy relations* (or, *syzygies*), see [19], [52]). For a small Maple package related to finite group invariants as well as generators (relations) of syzygy ideals, see `SP` package. [5]

4. PBW rings and algebras

There is a natural and important progression in developing the theory of Gröbner bases for Grassmann and Clifford algebras through the so called **left Poincaré-Birkhoff-Witt (PBW) rings and algebras**. While these rings are non-commutative, they possess a monomial basis and an admissible order can be defined on standard monomials. Furthermore, like ordinary polynomial rings $k[x_1, \ldots, x_n]$, they are domains and are left noetherian (Hilbert's Basis Theorem). Furthermore, PBW rings have the terminating multivariable division algorithm property, and every non-zero left ideal in a PBW ring possesses a Gröbner basis. In particular, if G is a Gröbner basis for a non-zero left ideal I in a PBW ring R, any "polynomial"

[3] Procedure `isContained` from SP package [5] is based on this proposition. It returns the polynomial g mentioned in the proposition.

$f \in R$ belongs to I if and only if the remainder on the division of f by G is zero. Hence, these rings have properties very similar to the ordinary polynomial rings.

Below we provide some basic definitions and examples of PBW rings. For an algorithmic theory of PBW rings and algebras including Gröbner bases computation see [17] on which our short introduction to these rings is based.

Definition 4.1. Let R be a ring containing a division ring k and let $x^\alpha = x_1^{\alpha_1} \dots , x_n^{\alpha_n}$ be a **standard term** where $x_1, \dots , x_n \in R$. The ring R is said to be **left polynomial** over k if the set $\{x^\alpha; \alpha \in \mathbb{N}^n\}$ is a basis of R as a left k-vectors pace. Then, every $f \in R$ has a **standard representation** $f = \sum_{\alpha\in\mathbb{N}^n} c_\alpha x^\alpha$.

Definition 4.2. An **admissible** order on $(\mathbb{N}^n, +)$ is a total order $\preceq$ satisfying the following two conditions: (i) $0 \prec \alpha,\ \forall \alpha \in \mathbb{N}^n$, and (ii) $\alpha+\gamma \prec \beta+\gamma,\ \forall \alpha, \beta, \gamma \in \mathbb{N}^n$ with $\alpha \prec \beta$.

For $0 \neq f \in R$, let $\exp(f) = \max_{\preceq}\{\alpha \in \mathbb{N}^n, c_\alpha \neq 0\}$ for an admissible order $\preceq$. We are now ready to define a PBW ring.

Definition 4.3. A ring R which is left-polynomial over k in $x_1, \dots , x_n$ is called a left **Poincaré-Birkhoff-Witt ring** (left PBW ring) if there exists an admissible order $\preceq$ on $(\mathbb{N}^n, +)$ that satisfies the following conditions:

- $\forall\, 1 \leq i < j \leq n,\ \exists\, q_{ij} \in k \setminus \{0\}$ s.t. $\exp(x_j x_i - q_{ji} x_i x_j) \prec \epsilon_i + \epsilon_j$ where $\epsilon_i = (0, \dots , 1, \dots , 0) \in \mathbb{N}^n$.
- $\forall\, 1 \leq i \leq n$ and $\forall\, a \in k \setminus \{0\},\ \exists\, q_{ja} \in k \setminus \{0\}$ s.t. $\exp(x_j a - q_{ja} x_j) \prec \epsilon_j$.

Let $p_{ji} = x_j x_i - q_{ji} x_i x_j$ for $1 \leq i < j \leq n$, and $p_{ja} = x_j a - q_{ja} x_j$ for $1 \leq i \leq n$ and $a \in k \setminus \{0\}$. We denote the left PBW ring R as $R = k\{x_1, \dots , x_n; Q, Q', \prec\}$ where $Q = \{x_j x_i = q_{ji} x_i x_j + p_{ji}; 1 \leq i < j \leq n\}$ and $Q' = \{x_j a = q_{ja} x_j + p_{ja}; 1 \leq j \leq n,\ a \in k^*\}$.

Definition 4.4. A left PBW ring R is called a **PBW algebra** if k is a commutative field and if $x_j a = a x_j$ for every $a \in k$ and $1 \leq j \leq n$.

We quote only two fundamental results proven in [17].

Lemma 4.5. *Any left PBW ring is a domain.*

Theorem 4.6 (Hilbert Basis Theorem). *Every left PBW ring is left noetherian.*

Here we list a few examples of PBW rings and algebras.

- Commutative polynomial ring $k[x_1, \dots , x_n]$: For every admissible order $\preceq$ on $\mathbb{N}^n$, we have
$$k[x_1, \dots , x_n] = k\{x_1, \dots , x_n;\ x_i x_j = x_j x_i,\ \preceq\}$$
is a PBW algebra.
- Let $\mathbf{g}$ be a finite-dimensional Lie k-algebra with k basis $\{x_1, \dots , x_n\}$. Let $\mathcal{U}(\mathbf{g})$ be its enveloping algebra. By the Poincaré-Birkhoff-Witt theorem, $\mathcal{U}(\mathbf{g})$ is left polynomial in $x_1, \dots , x_n$, and it is noetherian. In general, $\mathcal{U}(\mathbf{g}) = T(\mathbf{g})/I$,

where $T(\mathbf{g})$ is the tensor algebra over the linear space of $\mathbf{g}$ and I is a two-sided ideal generated by $x \otimes y - y \otimes x - [x,y]$, $\forall x, y \in \mathbf{g}$. Therefore, $\mathcal{U}(\mathbf{g})$ is a PBW algebra and

$$\mathcal{U}(\mathbf{g}) = k\{x_1, \ldots, x_n;\, x_i x_j = x_j x_i + [x_j, x_i],\, \preceq_{deglex}\}$$

- Let $\mathbf{q}$ be a multiplicatively anti-symmetric $n \times n$ matrix over k, i.e., $q_{i,j} \neq 0$ and $q_{i,j} = q_{j,i}^{-1}$ for all $1 \leq i, j \leq n$. The (multiparameter) n-**dimensional quantum space** $k_{\mathbf{q}}[x_1, \ldots, x_n]$ associated to $\mathbf{q}$ is the quotient of the free k-algebra $k\langle x_1, \ldots, x_n\rangle$ by the two-sided ideal associated to the relations $Q = \{x_j x_i = q_{ji} x_i x_j, j > i\}$. Let $\preceq$ be any admissible order on $\mathbb{N}^n$. Then

$$\mathcal{O}_{\mathbf{q}}(k^n) = k\{x_1, \ldots, x_n;\, Q, \preceq\}$$

  is a PBW algebra.
- There are constructive methods to obtain new (left) PBW rings as **Ore extensions** of a given (left) PBW ring. For example, skew polynomial Ore algebras and rings of differential operators are particular instances of the so called **iterated Ore extensions**.
- The n-th Weyl algebra $\mathbb{A}_n(k)$ is a PBW algebra.

Let R be a (left) PBW ring containing a division ring k as defined above. The multivariable division algorithm in R, the normal form of a polynomial f in R with respect to a set of polynomials F, the Gröbner bases in left-, and two-sided ideals computed through S-polynomials are all discussed at length in [17].

5. G-algebras and GR-algebras

In this section we describe associative algebras that possess PBW basis. These are G-algebras (first introduced by J. Apel [8]) and GR-algebras that have been studied at length in [35,36,38,39]. In particular, in [38,39] one can find a description of implementation of these algebras in Plural [50].

Definition 5.1. Let $\prec$ be a total well-ordering on $\mathbb{N}^n$. 1. Let A be an algebra with PBW basis and $\prec_A$ be an ordering on A induced by $\prec$. Then $\prec_A$ is a **monomial ordering** on A if the following conditions hold $\forall \alpha, \beta, \gamma \in \mathbb{N}^n$:

- If $x^\alpha \neq 0, x^\beta \neq 0$, then $\alpha \prec \beta \Rightarrow x^\alpha \prec x^\beta$,
- If $x^\alpha \prec x^\beta, x^{\alpha+\gamma} \neq 0$ and $x^{\beta+\gamma} \neq 0$ then $x^{\alpha+\gamma} \prec x^{\beta+\gamma}$.

2. Any $f \in A \setminus \{0\}$ can be written uniquely as $f = cx^\alpha + f'$, with $c \in k^*$ and $x^{\alpha'} \prec_A x^\alpha$ for any non-zero term $c' x^{\alpha'}$ of f'. Define $\mathrm{lm}(f) = x^\alpha$ as the **leading monomial** of f, and $\mathrm{lc}(f) = c$ as the **leading coefficient** of f.

A G-algebra is a defined as follows.

Definition 5.2. Let I be a two-sided ideal of $T = k\langle x_1, x_2, \ldots, x_n\rangle$ generated by the elements:

$$f_{j,i} = x_j x_i - c_{ij} x_i x_j - d_{ij}, \quad 1 \leq i < j \leq n, \quad c_{ij} \in k^*, \quad d_{ij} \in T. \tag{5.1}$$

A k-algebra $A = T/I = k\langle x_1, x_2, \dots, x_n \mid x_j x_i = c_{ij} x_i x_j + d_{ij}, \forall 1 \leq i < j \leq n\rangle$ is called a **G-algebra in n variables**, if the following conditions hold:

- **Ordering condition**: There exists a monomial well-ordering $\prec$ on T such that $\operatorname{lm}(d_{ij}) \prec x_i x_j, \forall 1 \leq i < j \leq n$.
- **Non-degeneracy condition**: $\forall 1 \leq i < j < k \leq n$, define polynomials

$$NDC_{ijk} = c_{ik}c_{jk} \cdot d_{ij}x_k - x_k d_{ij} + c_{jk} \cdot x_j d_{ik} - c_{ij} \cdot d_{ik} x_j + d_{jk} x_i - c_{ij} c_{ik} \cdot x_i d_{jk}$$

The condition is satisfied if all NDC_{ijk} reduce to 0 w.r.t. the relations (5.1).

In [36] Levandovskyy showed that a set of polynomials $F = \{f_{j,i}\}$ in (5.1) is a Gröbner basis for the two-sided ideal $I = \langle F\rangle$ with respect to the monomial well-ordering $\prec$ on T if an only if $\operatorname{NF}(NDC_{ijk}, F) = 0$. For defining two-sided Gröbner bases in free finitely generated algebras see also [10, 43].

The following is a summary of important properties of G-algebras due to Apel [8] and Levandovskyy [38].

Theorem 5.3. *Let A be a G-algebra in n variables.*

- *A has a PBW basis $\{x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n} \mid \alpha_k \in \mathbb{N}\}$.*
- *A is left and right Noetherian.*
- *A is an integral domain.*
- *A has a left and a right quotient ring.*

This last property is especially useful when defining GR-algebras. In particular recall the classical construction of Grassmann $\bigwedge V$ and Clifford algebras $C\ell(Q)$ as quotients of $T(V)/I$ of $T(V)$ - the ring of polynomials over k in **non-commuting variables** $X = \{x_1, \dots, x_n\}$ - over a suitable ideal $I \subset T(V)$. Here V is a free k-module with the basis X.

Definition 5.4 (Levandovskyy). Let B be a G-algebra and $I \subset B$ be a proper non-zero two-sided ideal. Then the quotient algebra B/I is called a **GR-algebra** (**Gröbner-ready algebra**).

Examples of G-algebras include the so called quasi-commutative polynomial rings, for example, the quantum plane and universal enveloping algebras of finite dimensional Lie algebras [17], positive (negative) parts of quantized enveloping algebras [34], Weyl algebras and their quantizations, Smith algebras, and some diffusion algebras [33]. Examples of GR-algebras include all G-algebras, Grassmann algebras and Clifford algebras $C\ell(Q)$ (Q may be degenerate), finite dimensional associative algebras given by structure constants [20]. For more examples and references see [38].

5.1. Gröbner bases in GR-algebras

For an introduction to left Gröbner bases in GR-algebra see [38]. As it was the case of ordinary polynomial rings and the PBW rings, one needs the concept of an admissible monomial order as in Definition 4.2, a left normal form to compute remainders on division, a concept of a left S-polynomials, a definition of a Gröbner basis, and algorithms for computing the latter.

Let $A^r = Ae_1 \oplus Ae_2 \oplus \cdots \oplus Ae_r$, where $e_i = (0, \ldots, 1, \ldots 0)$, be a free module with a monomial module ordering inherited from a monomial ordering in A. For a set $S \subset A^r$, define $\ell(S)$ to be a monoid generated by the leading exponents of elements of S, that is, $\ell(S) = \langle \alpha \,|\, \exists s \in S\,, \mathrm{LM}(s) = x^\alpha \rangle \subseteq \mathbb{N}^n$. Thus $\ell(S)$ is a monoid of leading exponents [17]. By Dickson's Lemma, $\ell(S)$ is finitely generated. Let $L(S) = \{x^\alpha \,|\, \alpha \in \ell(S)\}$ be a set of leading monomials of S.

This leads to the following definition of a Gröbner basis [38].

Definition 5.5. Let $I \subset A^r$ be a submodule. A finite set $G \subset I$ is called a **Gröbner basis** of I if and only if $L(G) = L(I)$, that is, for any $f \in I \setminus \{0\}$ there exists a $g \in G$ satisfying $\mathrm{LM}(g)|\mathrm{LM}(f)$. Here, $\mathrm{LM}(f)$ and $\mathrm{LC}(f)$ denote the leading monomial and the leading coefficient, respectively, of f with respect to the chosen monomial order (which are, in fact, sequences of leading monomials and coefficients when $r > 1$.)

Similarity with Definition 2.3 is obvious. In order to reduce a polynomial with respect to a list of polynomials, one needs the concept of a left normal form in A^r that Levandovskyy defines as follows.

Definition 5.6. Let $\mathcal{G}$ denote the set of all finite and ordered subsets $G \subset A^r$.

(i) A map $\mathrm{NF} : A^r \times \mathcal{G} \to A^r$, given by $(f, G) \mapsto \mathrm{NF}(f|G)$, is called a **(left) normal form** on A^r if, for all $f \in A^r, G \in \mathcal{G}$,
  (a) $\mathrm{NF}(f|G) \neq 0 \Rightarrow \mathrm{LM}(\mathrm{NF}(f|G)) \notin L(G)$,
  (b) $f - \mathrm{NF}(f|G) \in \langle G \rangle$.

(ii) Let $G = \{g_1, \ldots, g_s\} \in \mathcal{G}$. A representation $f = \sum_{i=1}^{s} a_i g_i$, $a_i \in A$ of $f \in \langle G \rangle$ satisfying $\mathrm{LM}(f) \geq \mathrm{LM}(a_i g_i)$ for all $i = 1, \ldots, s$ with $a_i g_i \neq 0$ is called a **standard (left) representation** of f (with respect to G).

The left normal form has been implemented in the TNB package [11] and is described below. The following lemma shows that the Gröbner basis defined for ideals in A^r solves the ideal membership problem. In fact, it is a bit more general as it solves the "submodule membership problem".

Lemma 5.7. *Let $I \subset A^r$ be a submodule, G a Gröbner basis of I and* $\mathrm{NF}(\cdot|G)$ *a normal form on A^r.*

(i) *For any $f \in A^r$ then $f \in I \Leftrightarrow \mathrm{NF}(f|G) = 0$.*

(ii) *If $J \subset A^r$ is a submodule with $I \subset J$, then $L(I) = L(J)$ implies $I = J$. In particular, G generates I as a left A-module.*

The S-polynomials are defined, as it can be expected, in a manner similar to the commutative case (see Definition 2.4).

Definition 5.8. Let $f, g \in A^r \backslash \{0\}$ with $\mathrm{LM}(f) = x^\alpha e_i$ and $\mathrm{LM}(g) = x^\beta e_j$, respectively. Set $\gamma = (\max(\alpha_1 \beta_1), \ldots, \max(\alpha_n \beta_n))$ and define the **left S-polynomial**

(when $r = 1$ or a vector of polynomials otherwise) of f and g as:

$$\mathrm{LeftSpoly}(f,g) = \begin{cases} x^{\gamma-\alpha}f - \dfrac{\mathrm{LC}(x^{\gamma-\alpha}f)}{\mathrm{LC}(x^{\gamma-\beta}g)}x^{\gamma-\beta}g, & \text{if } i=j, \\ 0, & \text{if } i \neq j. \end{cases}$$

The LeftSpoly form is needed for the Gröbner basis algorithm. A characterization of a Gröbner basis within a G-algebra can now be given. It will be the foundation for implementing a Gröbner basis algorithm.

Theorem 5.9. *Let $I \subset A^r$ be a left submodule and $G = \{g_1, \ldots, g_s\} \subset I$ and let* $\mathrm{LeftNF}(-|G)$ *be a left normal form on A^r with respect to G. The following are equivalent:*

(i) *G is a left Gröbner basis of I,*
(ii) $\mathrm{LeftNF}(f|G) = 0$ *for all $f \in I$,* $\boxed{\textit{Solves the Ideal Membership Problem}}$
(iii) *Each $f \in I$ has a left standard representation with respect to G,*
(iv) *G generates I as a left module and* $\mathrm{LeftNF}(\mathrm{LeftSpoly}(g_i, g_j)|G) = 0$ *for* $1 \leq i, j \leq s$.

In [39] one can find algorithms that compute LeftNF and a left Gröbner basis for a submodule I of a free module A^r. These algorithms are implemented in Plural [50] and TNB [11].

Remark 5.10.

- Theorem 5.9 implies that the algorithm based on that theorem to compute a Gröbner basis solves the ideal membership problem for left (right) ideals.
- The algorithm to compute a left Gröbner basis for a submodule I of a free module A^r generalizes the classical Buchberger's algorithm from $k[x_1, \ldots, x_n]$ to a free algebra in r variables.
- In order to perform computations in quotient GR-algebras, one needs to implement computation of two-sided Gröbner bases in ideals I. This is accomplished in yet another special algorithm in Plural [38]. This algorithm uses opposite algebra.
- Clifford algebra $C\ell_{p,q,d}$ is realized in Plural as a quotient of G-algebra $A = k\langle \mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n \,|\, \mathbf{e}_i\mathbf{e}_j = -\mathbf{e}_j\mathbf{e}_i,\ \forall j > i\rangle$ modulo a two-sided ideal $I \subset A$ generated as follows
$$I = \langle \{\mathbf{e}_i^2 + m_i \,|\, 1 \leq i \leq n,\ m_i \in \{-1, 0, 1\}\}\rangle$$
where for any constants m_i, the polynomials $\{\mathbf{e}_i^2 + m_i \,|\, 1 \leq i \leq n\}$ give a two-sided Gröbner basis in I. In particular, if we set $m_i = 0, 1 \leq i \leq n$, we get the Grassmann algebra [37].
- Clifford algebra $C\ell(B)$ of a general form B is not realized in Plural.

6. Grassmann and Clifford algebras in Plural

G-algebras are defined in Plural [50] using `ring` command extended to non-commutative variables. Then, a GR-algebra is defined as a quotient of a G-algebra modulo a two-sided ideal I. It is of the type `qring`, for example, `qring Q = twostd(I)`. There are various special-purpose libraries for pre-defined algebras. In particular, `clifford.lib` for Clifford algebras $C\ell(Q)$ and `nctools.lib` for non-commutative algebras including Grassmann algebra.

We give a few examples of computation of Gröbner bases. In a special case of a Grassmann algebra, definition of an admissible order is as follows.

Definition 6.1.
In Grassmann algebra, a monomial order $<$ is **admissible** if:
(1) $m > 1$ for every monomial m in the Grassmann basis;
(2) If $m_2 > m_1$ then $m_l \wedge m_2 \wedge m_r > m_l \wedge m_1 \wedge m_r$ for all monomials m_1, m_2, m_l, and m_r as long as $m_l \wedge m_2 \wedge m_r \neq 0$ and $m_l \wedge m_1 \wedge m_r \neq 0$.

It can be easily checked that the only admissible orders in Grassmann algebras are: `Lex`, `InvLex`, `Deg[Lex]`, and `Deg[InvLex]`.

Example 8. Here is a syntax to compute a Gröbner basis for the `Deg[Lex]` order in a left ideal $I = \langle 2\mathbf{e}_1 \wedge \mathbf{e}_2 + \mathbf{e}_2 - 4\mathbf{e}_3 \wedge \mathbf{e}_4, \mathbf{e}_1 \rangle$ in the Grassmann algebra $\bigwedge \mathbb{R}^4$. Procedure `Exterior` from Plural is used to set up the algebra. Notice that in Plural the wedge product $\wedge$ is just entered as $*$ with the software keeping track of the order of the arguments.[4]

```
LIB "nctools.lib";
ring R = 0,(e1,e2,e3,e4),dp;
def ER = Exterior();
setring ER;
ideal I =
2*e1*e2
+ e2
+ -4*e3*e4
,
e1
;
short=0;
option(redSB);
ideal GB = std(I);
write(":w C:/transferM/Out.txt",GB);
quit;
```

The Gröbner basis for I is $\{1\}$, hence the ideal I is the entire algebra.

---

[4]In the code below, the `write` line causes Plural to write its output into a file `Out.txt` that will later be read into Maple through the SINGULARPLURALlink package [2]. The line can be omitted when working directly with Plural.

Example 9. Consider an ideal $I = \langle 2\mathbf{e}_1 \wedge \mathbf{e}_2 + \mathbf{e}_2 - 4\mathbf{e}_3 \wedge \mathbf{e}_4, \mathbf{e}_1 \rangle \subset \bigwedge \mathbb{R}^4$. Plural and TNB return the following Gröbner basis for I in the `Deg[Lex]` order:

$$\{\mathbf{e}_2 \wedge \mathbf{e}_3,\, \mathbf{e}_2 \wedge \mathbf{e}_4,\, 4\mathbf{e}_3 \wedge \mathbf{e}_4 - \mathbf{e}_2,\, \mathbf{e}_1\} \tag{6.1}$$

Example 10. Consider polynomials $f_1 = \mathbf{e}_5 \wedge \mathbf{e}_6 - \mathbf{e}_2 \wedge \mathbf{e}_3$ and $f_2 = \mathbf{e}_4 \wedge \mathbf{e}_5 - \mathbf{e}_1 \wedge \mathbf{e}_3$ in $\bigwedge \mathbb{R}^6$. The Gröbner basis for the ideal $I = \langle f_1, f_2 \rangle$ in `Deg[Lex]` order returned by Plural and TNB is

$$\{\mathbf{e}_{145},\, \mathbf{e}_{245} + \mathbf{e}_{156},\, \mathbf{e}_{256},\, \mathbf{e}_{345},\, \mathbf{e}_{356},\, \mathbf{e}_{13} - \mathbf{e}_{45},\, \mathbf{e}_{23} - \mathbf{e}_{56}\} \tag{6.2}$$

where $\mathbf{e}_{145} = \mathbf{e}_1 \wedge \mathbf{e}_4 \wedge \mathbf{e}_5$, etc. This basis is different from the Gröbner GLB basis in Stokes (see below) for this ideal which is

$$\{\mathbf{e}_{56} - \mathbf{e}_{23},\, \mathbf{e}_{45} - \mathbf{e}_{13},\, \mathbf{e}_{234} + \mathbf{e}_{136},\, \mathbf{e}_{1236}\}. \tag{6.3}$$

Basis (6.2) is a GLIB basis in Stokes' terminology that solves the ideal membership problem while basis (6.3) is a GLB basis that does not solve that problem, hence it is different from (6.2).

Example 11. We compute a Gröbner basis in a left ideal $I = \langle \mathbf{e}_1 + 2\mathbf{e}_2, 3\mathbf{e}_1 + \mathbf{e}_1\mathbf{e}_2 \rangle$ in $C\ell_{2,0}$. The monomial order is `dp`= `Deg[Lex]`.

```
LIB "clifford.lib";
ring R = 0,(e1,e2),dp;
option(redSB);
option(redTail);
matrix M[2][2];
M[1,1]=2;M[2,2]=2;
clifAlgebra(M);
qring Q =twostd(clQuot);
ideal I =
e1
+ 2*e2
,
3*e1
+ e1*e2
;
short=0;
ideal GB = std(I);
```

The Gröbner basis for I is $\{1\}$, hence the ideal I is the entire algebra.

Example 12. Take $C\ell_{2,0} \cong \mathrm{Mat}(2,\mathbb{R})$ and a primitive idempotent $f = \frac{1}{2}(1 + \mathbf{e}_1)$. Let $S = C\ell_{2,0}f = \mathrm{span}_{\mathbb{R}}\{f, \mathbf{e}_2 f\}$ be a spinor ideal. Then a Gröbner basis for S in the monomial order `dp` = `Deg[Lex]` is $h = 1 + \mathbf{e}_1$. Note that $h = 2f$ is an almost idempotent.

Example 13. Consider $C\ell_{3,3} \cong \mathrm{Mat}_8(\mathbb{R})$ with a monomial order `Deg[InvLex]`. This order has no special name in Plural but we'll call it "degree inverse lex order" drp. It can be entered in Plural as (a(1:n),rp) where n refers to the number of non-commuting generators.

Algebra $C\ell_{3,3}$ is a simple algebra and any primitive idempotent f will be a product of three non-primitive idempotents [40]. In particular,

$$f = \tfrac{1}{2}(1+\mathbf{e}_{14})\tfrac{1}{2}(1+\mathbf{e}_{25})\tfrac{1}{2}(1+\mathbf{e}_{36}) \tag{6.4}$$

Then, a spinor left ideal $S = C\ell_{3,3}f \cong \mathbb{R}^8$ may be spanned by the following basis:

$$S = C\ell_{3,3}f = \mathrm{span}_{\mathbb{R}}\{f, \mathbf{e}_1 f, \mathbf{e}_2 f, \mathbf{e}_3 f, \mathbf{e}_{12} f, \mathbf{e}_{13} f, \mathbf{e}_{23} f, \mathbf{e}_{123} f\} \tag{6.5}$$

Let F be a list of the basis elements. A Gröbner basis for S returned by Plural contains this single polynomial

$$g = \mathbf{e}_{456} + \mathbf{e}_{156} - \mathbf{e}_{246} + \mathbf{e}_{126} + \mathbf{e}_{345} - \mathbf{e}_{135} + \mathbf{e}_{234} + \mathbf{e}_{123} \tag{6.6}$$

which is nilpotent $g^2 = 0$ and $\tilde{g} = -g$. Then g is related to the original idempotent f through a unit: $f = ug$ where $u = -\mathbf{e}_{123} = -u^{-1}$. Thus, $S = C\ell_{3,3}g = C\ell_{3,3}f$ as left ring ideals.

Furthermore, notice that the relation $f = ug$ implies $\tilde{f} = \tilde{g}\tilde{u} = (-g)(-u) = gu$ hence $\tilde{S} = \tilde{f}C\ell_{3,3} = gC\ell_{3,3}$. Thus g generates the left spinor space S as well as the right space $\tilde{S}$ of conjugate spinors.

Example 14. Take $C\ell_{3,1} \cong \mathrm{Mat}(4,\mathbb{R})$ and a primitive idempotent

$$f_1 = \tfrac{1}{4}(1+\mathbf{e}_1)(1+\mathbf{e}_{34}). \tag{6.7}$$

Let $S_1 = C\ell_{3,1}f_1 = \mathrm{span}_{\mathbb{R}}\{f_1, \mathbf{e}_2 f_1, \mathbf{e}_3 f_1, \mathbf{e}_{23} f_1\}$ be a spinor ideal. We set the same monomial order $\mathtt{dp} = \mathtt{Deg}[\mathtt{Lex}]$ in Plural.

Then, a Gröbner basis G for S_1 contains only one polynomial:

$$g = \mathbf{e}_{13} + \mathbf{e}_{14} - \mathbf{e}_3 - \mathbf{e}_4 = -\mathbf{e}_3 f_1, \quad g^2 = 0, \quad f_1 = -\mathbf{e}_3 g. \tag{6.8}$$

It is interesting to note that the idempotent f_1 does not constitute a Gröbner basis for S_1. Relations (6.8) imply that since f_1 and g differ by a unit, we get the same ideal $S_1 = C\ell_{3,1}f_1 = C\ell_{3,1}g$. Obviously, since G is a Gröbner basis for S_1, we have

$$\mathtt{NF}(f_1, G) = \mathtt{NF}(\mathbf{e}_2 f_1, G) = \mathtt{NF}(\mathbf{e}_3 f_1, G) = \mathtt{NF}(\mathbf{e}_{23} f_1, G) = 0.$$

So, for any spinor $s \in S_1$ we also get $\mathtt{NF}(s, G) = 0$. Now let $\tilde{f_1}$ be the reversion of f_1. So,

$$f_2 = \tilde{f_1} = \tfrac{1}{2}(1+\mathbf{e}_1)\tfrac{1}{2}(1-\mathbf{e}_{34}), \quad f_2 f_2 = f_2, \quad f_2 f_1 = f_1 f_2 = 0,$$

thus f_2 is a primitive idempotent and it generates another spinor ideal $S_2 = C\ell_{3,1}f_2$. Let $s_2 \in S_2$ then obviously $\mathtt{NF}(s_2, G) \neq 0$. For example, if

$$s_2 = (4 + 6\mathbf{e}_{23} - 2\mathbf{e}_{123})f_2 = \mathbf{e}_{123} - \mathbf{e}_{124} - \mathbf{e}_{134} + \mathbf{e}_{23} - \mathbf{e}_{24} - \mathbf{e}_{34} + \mathbf{e}_1 + 1 \in S_2$$

then $\mathtt{NF}(s_2, G) = -2\mathbf{e}_{124} - 2\mathbf{e}_{24} + 2\mathbf{e}_1 + 2 \neq 0$. This is of course the case since the Gröbner basis G solves the ideal membership problem for the ideal S_1 and $S_2 \cap S_1 = \{0\}$.

This last example shows the usefulness of the Gröbner basis for any left spinor ideal $S \subset C\ell_{p,q}$: Element $p \in C\ell_{p,q}$ belongs to S if and only if $\texttt{NF}(p, G) = 0$; that is, the remainder on division of p on G is zero where G is a Gröbner basis for S for some admissible monomial order.

Let's continue this last example.

Example 15. Take again $C\ell_{3,1} \cong \mathrm{Mat}(4, \mathbb{R})$ and a complete set of primitive mutually annihilating idempotents summing up to 1 :

$$f_1 = \tfrac{1}{4}(1 + \mathbf{e}_1)(1 + \mathbf{e}_{34}), \qquad f_2 = \tfrac{1}{4}(1 + \mathbf{e}_1)(1 - \mathbf{e}_{34}),$$
$$f_3 = \tfrac{1}{4}(1 - \mathbf{e}_1)(1 + \mathbf{e}_{34}), \qquad f_4 = \tfrac{1}{4}(1 - \mathbf{e}_1)(1 - \mathbf{e}_{34}). \tag{6.9}$$

Let $S_i = C\ell_{3,1} f_i$, $i = 1, 2, 3, 4$, be the corresponding left spinor spaces; that is, left minimal ideals in $C\ell_{3,1}$. A Gröbner basis in $\texttt{dp} = \texttt{Deg}[\texttt{Lex}]$ monomial order for each ideal is given, respectively for each ideal, by a single nilpotent element:

$$g_1 = \mathbf{e}_{13} + \mathbf{e}_{14} - \mathbf{e}_3 - \mathbf{e}_4, \qquad g_2 = \mathbf{e}_{13} - \mathbf{e}_{14} - \mathbf{e}_3 + \mathbf{e}_4,$$
$$g_3 = \mathbf{e}_{13} + \mathbf{e}_{14} + \mathbf{e}_3 + \mathbf{e}_4, \qquad g_4 = \mathbf{e}_{13} - \mathbf{e}_{14} + \mathbf{e}_3 - \mathbf{e}_4. \tag{6.10}$$

Let $u = -\frac{1}{4}\mathbf{e}_3$, hence u is a unit in $C\ell_{3,1}$. Due to the following set of identities,

$$f_1 = ug_1, \quad f_2 = ug_2, \quad f_3 = -ug_3, \quad f_4 = -ug_4 \tag{6.11}$$
$$\bar{f}_1 = g_1 u, \quad \bar{f}_2 = g_2 u, \quad \bar{f}_3 = -g_3 u, \quad \bar{f}_4 = -g_4 u \tag{6.12}$$

where $\bar{f}_1$ denotes conjugation in $C\ell_{3,1}$, we see that $S_i = C\ell_{3,1} f_i = C\ell_{3,1} g_i$, and $\bar{S}_i = \bar{f}_i C\ell_{3,1} = g_i C\ell_{3,1}$ since $\bar{g}_i = -g_i$, for $i = 1, 2, 3, 4$. Thus, these nilpotent elements g_i's can be used instead of the idempotents f_i's to generate left and right spinor ideals. Finally, we have this interesting way to factor each primitive idempotent into a product of nilpotents:

$$g_4 \bar{g}_1 = 16 f_1, \quad g_1 \bar{g}_4 = 16 f_4, \quad g_2 \bar{g}_3 = 16 f_3, \quad g_3 \bar{g}_2 = 16 f_2$$

with all other products $g_i g_j = 0$.

The last three examples may leave a false impression that a spinor ideal is always generated, as a left ideal in $C\ell_{p,q}$ considered as a ring, by a Gröbner basis consisting of one nilpotent generator. This last statement is true for these semisimple Clifford algebras: $C\ell_{2,1}, C\ell_{3,2}, C\ell_{4,3}$ yet it is not true for $C\ell_{0,7}$. In the next example we show that $C\ell_{0,7}$ is rather special in that any spinor ideal $S = C\ell_{0,7} f$ is generated by seven mutually annihilating nilpotent elements. Due to the importance of this algebra in triality [40, 44], we compute a Gröbner basis for a left spinor ideal in $C\ell_{0,7}$ and show its relation to a standard vector basis for S.

Example 16. Let $S = C\ell_{0,7} f$ where f is a primitive idempotent used by Lounesto (see [40, p. 307]). Thus,

$$f = \tfrac{1}{16}(1 + \mathbf{w})(1 - \mathbf{e}_{12\ldots7}) = \tfrac{1}{16}(1 - \mathbf{e}_{124})(1 - \mathbf{e}_{137})(1 - \mathbf{e}_{156})(1 - \mathbf{e}_{235}), \tag{6.13}$$

where $\frac{1}{2}(1 \pm \mathbf{e}_{12\ldots7})$ is a central idempotent and $\mathbf{w} = \mathbf{v}\mathbf{e}_{12\ldots7}^{-1}$. Element $\mathbf{v} \in \bigwedge^3 \mathbb{R}^{0,7}$ defines Fano triples and can be used to define octonionic product $a \circ b$ of two paravectors $a, b \in \mathbb{R} \oplus \mathbb{R}^{0,7}$ as shown by Lounesto. Element $\mathbf{v}$ is uniquely defined by the chosen primitive idempotent f as $\mathbf{v} = \mathbf{w}\mathbf{e}_{12\ldots7}$ and $\mathbf{w} = 8(f + \hat{f}) - 1$. [5] Here, $\mathbf{v} = \mathbf{e}_{124} + \mathbf{e}_{235} + \mathbf{e}_{346} + \mathbf{e}_{457} + \mathbf{e}_{561} + \mathbf{e}_{672} + \mathbf{e}_{713}$. A Gröbner basis for the ideal $S = C\ell_{0,7}f$ in monomial order `drp = Deg[InvLex]` is:

$$g_1 = \mathbf{e}_7 f,\ g_2 = \mathbf{e}_6 f,\ g_3 = \mathbf{e}_5 f,\ g_4 = \mathbf{e}_4 f,\ g_5 = \mathbf{e}_3 f,\ g_6 = \mathbf{e}_2 f,\ g_7 = \mathbf{e}_1 f. \tag{6.14}$$

It can be easily checked that $g_i g_j = 0$, $i, j = 1, \ldots, 7$. Units $\mathbf{e}_i, i = 1, \ldots, 7$, link these generators to the idempotent f. Of course, a Gröbner basis for the corresponding ideal $\hat{S} = C\ell_{0,7}\hat{f}$ is

$$\hat{g_1} = \mathbf{e}_7 \hat{f},\ \hat{g_2} = \mathbf{e}_6 \hat{f},\ \hat{g_3} = \mathbf{e}_5 \hat{f},\ \hat{g_4} = \mathbf{e}_4 \hat{f},\ \hat{g_5} = \mathbf{e}_3 \hat{f},\ \hat{g_6} = \mathbf{e}_2 \hat{f},\ \hat{g_7} = \mathbf{e}_1 \hat{f}, \tag{6.15}$$

and again $\hat{g_i}\hat{g_j} = 0$. It is interesting to note that all seven nilpotent polynomials (6.14) together with the idempotent f constitute eight basis elements for S considered as a vector space.

In our last example in this section we will list Gröbner bases G for left spinor ideals $S = C\ell_{p,q}f$ in all semisimple and simple Clifford algebras in dimensions $2 \leq p + q \leq 8$. Furthermore, we discuss only Clifford algebras $C\ell_{p,q}$ such that $fC\ell_{p,q}f \cong \mathbb{R}$, that is, when $p - q \mod 8 = 0, 1$, or 2. As our primitive idempotents f we will pick the ones listed in [1,3]. The Gröbner bases for the monomial order `drp = Deg[InvLex]` usually consist of one single nilpotent element, although there are exceptions. We will list also unit(s) (up to a non-zero constant) linking the Gröbner basis polynomial(s) with the generating idempotent f. That is, $f = u_i g_i$, $i = 1, \ldots, |G|$.

Example 17.
A. Semisimple algebras ($p - q = 1 \mod 4$).

- $S = C\ell_{2,1}f = \mathrm{LI}(g)$ where $g = \mathbf{e}_{13} + \mathbf{e}_{12} - \mathbf{e}_3 - \mathbf{e}_2$, $g^2 = 0$, and $u = \mathbf{e}_2$.
- $S = C\ell_{3,2}f = \mathrm{LI}(g)$ where $g^2 = 0$, $u = \mathbf{e}_{23}$, and

$$g = \mathbf{e}_{145} + \mathbf{e}_{125} - \mathbf{e}_{134} + \mathbf{e}_{123} + \mathbf{e}_{45} + \mathbf{e}_{25} - \mathbf{e}_{34} + \mathbf{e}_{23}. \tag{6.16}$$

- (Exception) $S = C\ell_{0,7}f = \mathrm{LI}(g_1, \ldots, g_7)$ where $f = u_i g_i$, $g_i g_j = 0$, $1 \leq i, j \leq 7$, and the units are $\mathbf{e}_7, \mathbf{e}_6, \mathbf{e}_5, \mathbf{e}_4, \mathbf{e}_3, \mathbf{e}_2, \mathbf{e}_1$.
- $S = C\ell_{4,3}f = \mathrm{LI}(g)$ where $g^2 = 0$, $u = \mathbf{e}_{234}$, and

$$\begin{aligned} g = \mathbf{e}_{1567} &+ \mathbf{e}_{1267} - \mathbf{e}_{1357} + \mathbf{e}_{1237} + \mathbf{e}_{1456} - \mathbf{e}_{1246} + \mathbf{e}_{1345} + \mathbf{e}_{1234} \\ &- \mathbf{e}_{567} - \mathbf{e}_{267} + \mathbf{e}_{357} - \mathbf{e}_{237} - \mathbf{e}_{456} + \mathbf{e}_{246} - \mathbf{e}_{345} - \mathbf{e}_{234}. \end{aligned} \tag{6.17}$$

B. Simple algebras when $p - q \neq 1 \mod 4$.

- $S = C\ell_{1,1}f = \mathrm{LI}(g)$ where $g = \mathbf{e}_1 + \mathbf{e}_2$, $g^2 = 0$, and the unit $u = \mathbf{e}_1$.
- (Exception) $S = C\ell_{2,0}f = \mathrm{LI}(g)$ where $g = 1 + \mathbf{e}_1$, $g^2 = 2g$, and $u = 1$.
- $S = C\ell_{2,2}f = \mathrm{LI}(g)$ where $g = \mathbf{e}_{34} + \mathbf{e}_{14} - \mathbf{e}_{23} + \mathbf{e}_{12}$, $g^2 = 0$, and $u = \mathbf{e}_{12}$.
- $S = C\ell_{3,1}f = \mathrm{LI}(g)$ where $g = \mathbf{e}_{13} + \mathbf{e}_{14} - \mathbf{e}_3 - \mathbf{e}_4$, $g^2 = 0$, and $u = \mathbf{e}_3$.

---

[5] Of course, the choice of Fano triples decides the form of $\mathbf{v}$ which in turn decides which primitive idempotent f is chosen to generate the spinor ideal.

- (Exception) $S = C\ell_{0,6}f = \mathrm{LI}(g_1,\dots,g_5)$ where $f = u_i g_i$, $g_i g_1 \neq 0$, $g_i g_j = 0$ for $1 \leq i \leq 5$, $2 \leq j \leq 5$, $g_1^2 = g_1$, $g_i^2 = 0, i = 2,\dots,5$, and the units are $1, \mathbf{e}_1, \mathbf{e}_3, \mathbf{e}_5, \mathbf{e}_6$.
- $S = C\ell_{3,3}f = \mathrm{LI}(g)$ where $g^2 = 0$, $u = \mathbf{e}_{123}$, and
$$g = \mathbf{e}_{456} + \mathbf{e}_{156} - \mathbf{e}_{246} + \mathbf{e}_{126} + \mathbf{e}_{345} - \mathbf{e}_{135} + \mathbf{e}_{234} + \mathbf{e}_{123}. \tag{6.18}$$
- $S = C\ell_{4,2}f = \mathrm{LI}(g)$ where $g^2 = 0$, $u = \mathbf{e}_{34}$, and
$$g = \mathbf{e}_{156} + \mathbf{e}_{136} - \mathbf{e}_{145} + \mathbf{e}_{134} + \mathbf{e}_{56} + \mathbf{e}_{36} - \mathbf{e}_{45} + \mathbf{e}_{34}. \tag{6.19}$$
- (Exception) $S = C\ell_{0,8}f = \mathrm{LI}(g_1,\dots,g_7)$ where
$$f = \tfrac{1}{16}(1+\mathbf{e}_{123})(1+\mathbf{e}_{345})(1+\mathbf{e}_{146})(1+\mathbf{e}_{367}),$$
$f = u_i g_i$, $g_i g_j = 0$, for $1 \leq i, j \leq 7$, and the units are $\mathbf{e}_7, \mathbf{e}_6, \mathbf{e}_5, \mathbf{e}_4, \mathbf{e}_3, \mathbf{e}_2, \mathbf{e}_1$.
- (Exception) $S = C\ell_{1,7}f = \mathrm{LI}(g_1,\dots,g_5)$ where $f = u_i g_i$, $g_i g_j = 0$, for $1 \leq i, j \leq 5$, and the units are $\mathbf{e}_1, \mathbf{e}_{14}, \mathbf{e}_{12}, \mathbf{e}_{17}, \mathbf{e}_{16}$.
- $S = C\ell_{4,4}f = \mathrm{LI}(g)$ where $g^2 = 0$, $u = \mathbf{e}_{1234}$, and
$$\begin{aligned} g = {}& \mathbf{e}_{5678} + \mathbf{e}_{1678} - \mathbf{e}_{2578} + \mathbf{e}_{1278} + \mathbf{e}_{3568} - \mathbf{e}_{1368} + \mathbf{e}_{2358} + \mathbf{e}_{1238} \\ & - \mathbf{e}_{4567} + \mathbf{e}_{1467} - \mathbf{e}_{2457} - \mathbf{e}_{1247} + \mathbf{e}_{3456} + \mathbf{e}_{1346} - \mathbf{e}_{2345} + \mathbf{e}_{1234}. \end{aligned} \tag{6.20}$$
- $S = C\ell_{5,3}f = \mathrm{LI}(g)$ where $g^2 = 0$, $u = \mathbf{e}_{345}$, and
$$\begin{aligned} g = {}& \mathbf{e}_{1678} + \mathbf{e}_{1378} - \mathbf{e}_{1468} + \mathbf{e}_{1348} + \mathbf{e}_{1567} - \mathbf{e}_{1357} + \mathbf{e}_{1456} + \mathbf{e}_{1345} \\ & - \mathbf{e}_{678} - \mathbf{e}_{378} + \mathbf{e}_{468} - \mathbf{e}_{348} - \mathbf{e}_{567} + \mathbf{e}_{357} - \mathbf{e}_{456} - \mathbf{e}_{345}. \end{aligned} \tag{6.21}$$
- (Exception) $S = C\ell_{8,0}f = \mathrm{LI}(g_1,\dots,g_7)$ where
$$f = \tfrac{1}{16}(1+\mathbf{e}_{1})(1+\mathbf{e}_{2345})(1+\mathbf{e}_{4567})(1+\mathbf{e}_{2468}),$$
$f = u_i g_i$, $g_i g_1 \neq 0$, for $1 \leq i \leq 7$, $g_i g_j = 0$ for $1 \leq i \leq 7$, $2 \leq j \leq 7$, $g_1^2 = g_1$, $g_i^2 = 0, i = 2,\dots,7$, and the units are $1, \mathbf{e}_2, \mathbf{e}_5, \mathbf{e}_7, \mathbf{e}_8, \mathbf{e}_{23}, \mathbf{e}_{24}$.

Let us summarize our findings. As above, $S = C\ell_{p,q}f$ is a left spinor ideal.

1. In all cases other than the Exceptions, S has a Gröbner basis consisting of one nilpotent element.
2. In $C\ell_{0,7}$ and $C\ell_{0,8}$, the ideal S has a Gröbner basis consisting of seven mutually annihilating nilpotents.
3. In $C\ell_{1,7}$, the left ideal S has a Gröbner basis consisting of five mutually annihilating nilpotents.
4. In $C\ell_{0,6}$, the Gröbner basis for S has five elements: one idempotent and four nilpotents.
5. In $C\ell_{8,0}$, the Gröbner basis for S has seven elements: one idempotent and six nilpotents.
6. In $C\ell_{2,0}$, the only Gröbner basis polynomial is the generating idempotent.
7. In every case, each Gröbner basis polynomial is related via a different unit to the generating idempotent f.

The reason(s) for these exceptions should be explained.

7. GLB and GLIB bases in Grassmann algebras

In 1990 Timothy Stokes showed that Grassmann algebra is suitable for algorithmic treatment when treated as graded-commutative algebra of "exterior polynomials". In [51] he defined two different Gröbner bases for left ideals in **(super) Grassmann polynomial algebra** $\bigwedge_{n,m}$ of order (n,m) over a field k, $\operatorname{char} k \neq 2$. Furthermore, he proved convergence of his algorithms for computing Gröbner Left Bases (GLB) and Gröbner Left Ideal Basis (GLIB) for such ideals.

Stokes showed that obtaining Gröbner bases in Grassmann algebras is more complicated than in PBW algebras, which are domains, due to the abundance of zero divisors. This led him to the two types of Gröbner bases. Such dichotomy of bases does not exist in PBW algebras. In this section we will briefly summarize Stokes' approach as it is very instructive in gaining a better understanding of the computational differences.

Definition 7.1 (Stokes). The **(super) Grassmann polynomial algebra** $\bigwedge_{n,m}$ of order (n,m) over a field k, $\operatorname{char} k \neq 2$, is the associative algebra $k[\alpha_1,\dots,\alpha_m] \otimes \bigwedge V$ where $V = \operatorname{span}_k\{\mathbf{e}_1,\dots,\mathbf{e}_n\}$.

Stokes proved that $\bigwedge_{n,m}$ is left noetherian and, hence, every left ideal has a finite basis. Thus, in particular, the Grassmann algebra $\bigwedge_n$ (identified with $\bigwedge_{n,0}$) is left noetherian. Any product p of variables (a monomial) in $\bigwedge_{n,m}$ can be written in a unique canonical way by applying the commuting and anticommuting rules above once we select a linear order, for example, $\mathbf{e}_1 < \mathbf{e}_2 < \cdots < \mathbf{e}_n < \alpha_1 < \alpha_2 < \cdots < \alpha_m$. Following Stokes, let $\overline{p}$ denote the unique corresponding element in the set of monomials and let $T_{n,m}$ be the set of those monomials for which $p = \overline{p}$ (Stokes calls them *positive*).

In order to define a Gröbner basis, as we have seen above, one needs to define an *admissible* monomial order on $T_{n,m}$, a *reduction relation* that will be noetherian (so that a sequence of reductions of any $f \in \bigwedge_{n,m}$ would terminate after a finite number of steps), a definition of a Gröbner basis, and an algorithm to compute such basis that will terminate.

Definition 7.2 (Stokes). A total ordering $\leq$ on $T_{n,m}$ is **compatible** (or, **admissible**) if, for all $s,t,u \in T_{n,m}$, (i) $1 < t$, and (ii) If $s < t$ then $\overline{us} < \overline{ut}$, as long as $us \neq 0$ and $ut \neq 0$ (and, automatically, $\overline{su} < \overline{tu}$ by graded-commutativity).[6]

One such order is the 'total degree order' (or, `Deg[InvLex]`) on $\bigwedge_3$:

$$\mathbf{e}_{123} > \mathbf{e}_{23} > \mathbf{e}_{13} > \mathbf{e}_{12} > \mathbf{e}_3 > \mathbf{e}_2 > \mathbf{e}_1 > 1$$

and another is `Deg[Lex]` which gives:

$$\mathbf{e}_{123} > \mathbf{e}_{12} > \mathbf{e}_{13} > \mathbf{e}_{23} > \mathbf{e}_1 > \mathbf{e}_2 > \mathbf{e}_3 > 1$$

[6]The condition $1 < t$ for all $t \in T_{n,m}$ is necessary to guarantee termination of the division algorithm. See also [43].

Proposition 7.3 (Stokes). *Every compatible order on $T_{n,m}$ is a well-order, i.e., every non-empty subset of $T_{n,m}$ has a smallest element w.r.t. to the order.*

One can now define, in a standard way, $\mathrm{LTerm}(f)$, $\mathrm{LCoeff}(f)$, $\mathrm{LMon}(f)$, and $\mathrm{coef}(f,t)$ for any polynomial $f \in \bigwedge_{n,m}$ and term t of f.

Definition 7.4 (Stokes). Let F be a finite list of polynomials. Then, polynomial g **left reduces to** h **modulo** F, denoted $g \rightarrow_F h$, if there exist an $u \in T_n$ and $f \in F$ with $h = g - buf$, and

$$\mathrm{coef}(g, \overline{u \cdot \mathrm{LTerm}(f)}) \neq 0, \quad b = \mathrm{coef}(g, \overline{u \cdot \mathrm{LTerm}(f)})/\mathrm{LCoeff}(f) \cdot \mathrm{sign}$$

where and $\mathrm{sign} = \mathrm{coef}(u \cdot \mathrm{LTerm}(f), \overline{u \cdot \mathrm{LTerm}(f)})$. Here $\rightarrow_F$ is the **reduction relation** for F.

As a consequence of Proposition 7.3, Stokes proved that the reduction relation $\rightarrow_F$ is a noetherian relation; that is, there is no infinite sequence of reductions of any $f \in \bigwedge_{n,m}$. Thus, h will be a **normal form** of f modulo F if there is a finite sequence of reductions

$$f \rightarrow_F f_1 \rightarrow_F f_2 \rightarrow_F \cdots \rightarrow_F h$$

yet $\nexists\, h' \in \bigwedge_{n,m}$ such that $h \rightarrow_F h'$. We will the write $\mathrm{NF}(f,F) = h$ for the given admissible order $<$.

Requiring $\rightarrow_F$ to be *confluent* and terminating reduction relation leads to the idea of a GLB basis for left ideals whereas requiring that any polynomial in the left ideal generated by the basis reduces to zero modulo the basis leads to the idea of a GLIB basis. Recall the following definition of reduction relations from [13, 14, 51].

Definition 7.5. Let $\leftrightarrow_F^*$ be the equivalence relation defined by $\rightarrow_F$, that is, $\leftrightarrow_F^*$ is the *reflexive, symmetric, and transitive closure* of $\rightarrow_F$. Let $\rightarrow_F^*$ be the *reflexive transitive closure* of $\rightarrow_F$. Reduction $\rightarrow_F$ on a set S is **confluent** if for all $x, y, z, \in S$, whenever $z \rightarrow_F^* x$ and $z \rightarrow_F^* y$, there is $w \in S$ such that $x \rightarrow_F^* w$ and $y \rightarrow_F^* w$. We say $f \equiv_F^l g$ (f is *left-congruent to* g *modulo* F) whenever $f - g \in \mathrm{LI}(F)$ and $f \equiv_F g$ (f is *congruent to* g *modulo* F) whenever $f - g \in \mathrm{Ideal}(F)$.[7]

A *Gröbner left basis* (GLB) basis for a left ideal $\mathrm{LI}(F)$ is defined by Stokes as follows.

Definition 7.6. F is a **Gröbner left basis (GLB)** for a left ideal $\mathrm{LI}(F)$ if for all g, h_1 and h_2 in $\bigwedge_{n,m}$, if h_1 and h_2 are normal forms of g modulo F, then $h_1 = h_2$.

Thus, a GLB basis F gives uniqueness of the normal form when reducing any polynomial modulo the basis, but, as we will see, it does not solve the left ideal membership problem. The following theorem due to Stokes [51] characterizes a GLB basis and gives a foundation for his algorithm to compute it.

[7] $\mathrm{LI}(F)$ (resp., $\mathrm{Ideal}(F)$) denotes the left ideal (resp., ideal) in $\bigwedge_{n,m}$ generated by F.

Theorem 7.7 (GLB Characterization Theorem). *The following conditions are equivalent.*

(i) *F is a GLB.*
(ii) *If $f_1, f_2 \in F$, and $t \in T_{n,m}$ satisfies $t \cdot \mathrm{lcm}(\mathrm{LMon}(f_1), \mathrm{LMon}(f_2)) \neq 0$, then $t \cdot S(f_1, f_2) \rightarrow_F^* 0$ where $S(f_1, f_2)$ is an S-polynomial of f_1, f_2.*[8]

Note the important difference with the commutative case: It is not just $S(f_1, f_2) \rightarrow_F^* 0$ that is required but also $t \cdot S(f_1, f_2) \rightarrow_F^* 0$. This is because $f \rightarrow_F^* 0$ does not imply $t \cdot f \rightarrow_F^* 0$ for all terms $t \in T_{n,m}$ due to the presence of zero divisors in $\bigwedge_{n,m}$.

Example 18. Let $f_1 = \mathbf{e}_{2456} - \mathbf{e}_3$, $f_2 = \mathbf{e}_{14} - \mathbf{e}_1$, $f_3 = \mathbf{e}_3 \in \bigwedge_6$ and let the admissible monomial order be `Deg[InvLex]`. A GLB basis G for the left ideal $\mathrm{LI}(f_1, f_2, f_3)$ required computation of six S-polynomials and is as follows:

$$f_1 = \mathbf{e}_{2456} - \mathbf{e}_3, \quad f_2 = \mathbf{e}_{14} - \mathbf{e}_1, \quad f_3 = \mathbf{e}_3, \quad f_4 = -\mathbf{e}_{13} + \mathbf{e}_{1256}. \tag{7.1}$$

It can be easily demonstrated that even if $f \in \mathrm{LI}(f_1, f_2, f_3) = \mathrm{LI}(G)$, the remainder $\mathrm{NF}(f, G)$ need not be zero. For example, letting

$$\begin{aligned} f &= f_1 + (-6 - 6\mathbf{e}_{24} + \mathbf{e}_{16} - 6\mathbf{e}_3 + \mathbf{e}_{35}) f_2 + \\ &\qquad (-2 + 4\mathbf{e}_3 - 6\mathbf{e}_5 - 6\mathbf{e}_{1346}) f_3 + (\mathbf{e}_{126} + \mathbf{e}_{24}) f_4 \\ &= -3\mathbf{e}_3 + \mathbf{e}_{2456} - 6\mathbf{e}_{13} - 6\mathbf{e}_{14} + 6\mathbf{e}_1 + \\ &\qquad 6\mathbf{e}_{124} + 6\mathbf{e}_{134} - \mathbf{e}_{135} - \mathbf{e}_{1345} + 6\mathbf{e}_{35} + \mathbf{e}_{1234} \end{aligned} \tag{7.2}$$

we find that $\mathrm{NF}(f, G) = -2\mathbf{e}_3 + 6\mathbf{e}_{12}$ even though $\mathrm{NF}(f_i, G) = 0$, $i = 1, \ldots, 4$.

It should be observed that the GLB Characterization Theorem:

- Does not guarantee that the GLB basis G produced by the algorithm based on Theorem 7.7 is unique, minimal, or reduced.
- $\mathrm{NF}(S(f_i, f_j), G) = 0$, $f_i \neq f_j$, $f_i, f_j \in G$ like for any Gröbner basis in a commutative case, but also $\mathrm{NF}(t \cdot S(f_i, f_j), G) = 0$, $f_i \neq f_j$, $f_i, f_j \in G$ but only for $t \in T_{n,m}$ such that $t \cdot \mathrm{lcm}(\mathrm{LMon}(f_i), \mathrm{LMon}(f_j)) \neq 0$.
- Does not guarantee that $\mathrm{NF}(t \cdot S(f_1, f_2), G) = 0$, $f_1, f_2 \in G$, for *every* Grassmann basis monomial t. However, it is possible to include such extra non-zero remainders in G and a new set is also a GLB basis.

A *Gröbner left ideal basis* is defined as follows.

Definition 7.8 (Stokes). F is a **Gröbner left ideal basis (GLIB)** for $\mathrm{LI}(F)$ if $f \in \mathrm{LI}(F)$ implies that $f \rightarrow_F^* 0$.

Stokes showed that every GLIB Gröbner basis is a GLB Gröbner basis and proved the following useful characterization theorem.

Theorem 7.9 (GLIB Characterization Theorem). *The following statements are equivalent.*

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[8] The S-polynomial $S(f_1, f_2)$ is defined in a similar manner as we have seen it before. See [51] for details.

(i) *F is a GLIB.*
(ii) *If $f_1, f_2 \in F$, then for any $t \in T_{n,m}$, $t \cdot f_1 \rightarrow_F^* 0$ and $t \cdot \mathrm{LeftSpoly}(f_1, f_2) \rightarrow_F^* 0$.*
(iii) *For $f_1, f_2 \in F$ and for any $t_1, t_2 \in T_{n,m}$ satisfying the conditions*

$$t_1 \cdot \mathrm{LTerm}(f_1) = 0 \text{ and } t_2 \cdot \mathrm{lcm}(\mathrm{LTerm}(f_1), \mathrm{LTerm}(f_2)) \neq 0,$$

then $t_1 \cdot f_1 \rightarrow_F^* 0$ and $t_2 \cdot \mathrm{LeftSpoly}(f_1, f_2) \rightarrow_F^* 0$.

Remark 7.10. See Stokes [51] for a discussion of correctness and termination of his algorithm to compute a GLIB basis G from the initial list F. Stokes proves that $\mathrm{LI}(F) = \mathrm{LI}(G)$.

Example 19. Let f_1, f_2, f_3 be as in Example 18. A GLB basis for $\mathrm{LI}(f_1, f_2, f_3)$ was shown in (7.1). A GLIB basis G_1 for $\mathrm{LI}(f_1, f_2, f_3)$ for the same monomial order `Deg[InvLex]` required computation of six S-polynomials and is given by

$$g_1 = \mathbf{e}_{2456} - \mathbf{e}_3, \quad g_2 = \mathbf{e}_{14} - \mathbf{e}_1, \quad g_3 = \mathbf{e}_3, \quad g_4 = \mathbf{e}_1 \tag{7.3}$$

This time, we find that $\mathrm{NF}(f, G_1) = 0$ where f is given in (7.2).

For a more detailed presentation of algorithms that compute GLB and GLIB bases as well as for more detailed step-by-step examples see [11, 12].

8. Conclusions

Here are some computational differences and similarities when computing Gröbner bases in $k[x_1, \ldots, x_n]$, and Grassmann and Clifford algebras:

- $k[x_1, x_2, \ldots, x_n]$ is a domain for any field k–in fact, it is a unique factorization domain (UFD). In particular, it has no non-zero zero divisors. Grassmann algebras are never domains whereas most Clifford algebras $C\ell(Q)$ are not domains either as they possess non-trivial idempotents $e^2 = e$, $e \neq 0, 1$, and $e(e-1) = 0$.
- Let $R = k[x_1, \ldots, x_n]$ and k be a field. Then, R is a noetherian ring. In particular, every ideal in R is finitely generated, equiv., R has ACC, equiv., R satisfies the maximum condition: Every non-empty family $\mathcal{F}$ of ideals in R has a maximal element. Any quotient ring R/I where I is any ideal, is also noetherian.
- Grassmann algebras and superalgebras are left noetherian [51].
- Left and right ideals in Grassmann and Clifford algebras do not coincide due to non-commutativity whereas they are identical in $k[x_1, \ldots, x_n]$.
- In the polynomial ring $k[x_1, \ldots, x_n]$ the division algorithm terminates due to noetherianness of the ring. Some non-commutative algebras are not noetherian [43]; therefore, the division algorithm may not terminate in general. However, Grassmann algebra is left noetherian as it has no infinite ascending chain of ideals [46, 51].

- When reducing an S-polynomial $S(f_i, f_j) \in R = k[x_1, \ldots, x_n]$ modulo a finite set of polynomials F, for example, when computing a Gröbner basis, suppose $\overline{S(f_i, f_j)}^F = 0$. Then, $\overline{m \cdot S(f_i, f_j)}^F = 0$ for any monomial $m = x^\alpha \in R$. This is often not the case in Grassmann or Clifford algebra due to the presence of non-zero zero divisors. This complicates computation of Gröbner bases in these algebras.

A major difference aside from the non-commutativity when computing Gröbner bases in Grassmann and Clifford algebras is the presence, if not abundance, of non-zero zero divisors. Algorithms to compute a left normal form and then a left Gröbner basis in [35, 36] generalize the classical Buchberger's algorithm from $k[x_1, \ldots, x_n]$ to a quotient GR-algebras and solve the ideal membership problem. However, care must be taken as vanishing of an S-polynomial modulo a set of "polynomials" in Grassmann or Clifford algebra does not guarantee its vanishing when the S-polynomial is pre-multiplied by a basis monomial. This has been shown clearly by Stokes [51] who has introduced two types of Gröbner bases in left ideals in Grassmann algebra: a GLB basis which guarantees uniqueness of a the remainder, and GLIB which also guarantees that $\overline{f}^G = 0$ modulo a GLIB-type Gröbner basis G is equivalent to $f \in \langle G \rangle$. See [11] for implementation of GLB and GLIB bases for Grassmann algebras in a Maple package `TNB`.

Finally, we mention that non-commutative Gröbner bases in Grassmann algebras and the issue of ideal membership surface when analyzing systems of partial differential equations that arise in physics, i.e., in exterior differential systems as shown in [29] and references therein. In particular, Hartley and Tuckey provide another approach through the so called *saturating sets* to Gröbner bases in Grassmann and Clifford algebras in a REDUCE package called `XIDEAL`. A major application emphasized in the paper is that Gröbner bases may help simplify exterior differential systems and so help solve systems of partial differential equations.

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