DEPARTMENT OF MATHEMATICS

TECHNICAL REPORT

SCHUR POLYNOMIALS AND THE IRREDUCIBLE REPRESENTATIONS OF $\mathcal{S}_n$

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APRIL 2009

No. 2009-2

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Schur Polynomials and the Irreducible Representations of $\mathcal{S}_n$*

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April 15, 2009

Abstract

One of the main problems in representation theory is the decomposition of a group representation into irreducible components. The Littlewood-Richardson rule gives a combinatorial method to determine the coefficients of irreducibles of the tensor product representation of a general linear group. The same method can be used to find the coefficients of irreducible modules, called Specht modules, of the induced representation of a symmetric group. The Littlewood-Richardson rule uses Young tableaux and Schur polynomials to calculate the desired coefficients. Indeed the Schur polynomials are in direct correspondence with the conjugacy classes of $\mathcal{S}_n$. Young tableaux are also discussed, and the Littlewood-Richardson rule is introduced along with examples to illustrate. [1]

Keywords: Schur polynomial, symmetric group, Littlewood-Richardson rule, Young tableau, irreducible representation, character, symmetric polynomial

*Submitted in partial fulfillment of requirements in MATH 6991, fall 2008

[1]AMS Subject Classification: Primary 05E05, 20C30; Secondary 05E10

Contents

1 Introduction

Schur polynomials are certain homogeneous symmetric polynomials in n indeterminates with integer coefficients and correspond to the irreducible representations of $\mathcal{S}_n$. One of the main problems in the field of representation theory is the decomposition of a representation into irreducible components realized as irreducible modules.

The Littlewood-Richardson rule can be used to find the irreducible modules of the symmetric group, or in a general linear group it can be used to find the decomposition of a tensor product into irreducibles, in both cases by looking at the corresponding Schur functions. An overview of the representation of the symmetric group will be presented, but the focus of this paper is on the calculation of Littlewood-Richardson coefficients using the Littlewood-Richardson rule.

Throughout this paper we denote a partition λ of positive integer n by $\lambda \vdash n$, where λ is a sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ and the λ_i are weakly decreasing positive nonzero integers and $\sum_{i=i}^{l} \lambda_i = n$. The size of λ is denoted $|\lambda|$ and in general $|\lambda| = n$.

The symmetric group $\mathcal{S}_n$ is referenced throughout the paper, so we give a brief introduction here.

Definition 1. *The symmetric group, denoted $\mathcal{S}_n$, is the group consisting of all bijections from $\{1, 2, \ldots, n\}$ to itself under the operation of composition. The elements of $\mathcal{S}_n$ are permutations, which we multiply from right to left. That is, $\pi\sigma$ means to apply permutation σ first and then apply permutation π.*

The notation we will use for the permutations of $\mathcal{S}_n$ is **cycle notation**. For example, take the permutation $\pi = (142)(36)(5)$. This notation tells us that the permutation π maps 1 to 4, 4 to 2, 2 to 1, 3 to 6, and 5 to itself. Notice that π in our example consists of 3 disjoint cycles, and since disjoint cycles commute, reordering the cycles does not change the permutation. That is, $(142)(36)(5) = (36)(142)(5) = (5)(36)(142)$.

Definition 2. *A* ***k-cycle*** *or* ***cycle of length k****, is a cycle containing k elements.*

For example, $\pi = (142)(36)(5)$ contains a 3-cycle, 2-cycle, and a 1-cycle. We will find that cycles of particular lengths, or cycle types, determine conjugacy classes of the symmetric group $\mathcal{S}_n$. Since these conjugacy classes are intimately connected with Schur polynomials, we will introduce Schur polynomials before discussing the symmetric group further. The definitions, theorems, and propositions in this paper are taken from [2] unless otherwise indicated. Some proofs have been omitted, but they can be found in [2] as well.

2 Schur Polynomials

Schur polynomials, also called Schur functions, arise in many different contexts. These polynomials form a basis for the space of all symmetric polynomials. There are different ways to define Schur polynomials. A Schur polynomial depends on a partition λ of a positive integer d. One definition of these polynomials as they arise from antisymmetric polynomials will be presented later in this paper. For now we introduce Schur polynomials using a combinatorial approach with Young tableaux, which also play an important role in the Littlewood-Richardson rule. We will see that Young tableaux depend on a given partition λ as well, and hence there is a natural relationship between Schur polynomials and Young tableaux.

2.1 Young Tableaux

Definition 3. *Suppose $\lambda = (\lambda_1, \ldots, \lambda_l) \vdash n$ where $l \geq 1$. The* ***Ferrers diagram*** *of shape λ is an array of n squares having l left-justified rows with row i containing λ_i squares.*

Example 1. The partition $\lambda = (3, 2, 2, 1)$ has Ferrers diagram

$$\begin{array}{|c|c|c|}\hline & & \\ \hline & \\ \cline{1-2} & \\ \cline{1-2} \\ \cline{1-1}\end{array}$$

Definition 4. *A **filling** of a Ferrers diagram is any way of putting a positive integer in each box of the diagram. Let $\mu = (\mu_1, \mu_2, \ldots, \mu_k)$ be a filling of a Ferrers diagram. Each μ_i is the number of times the integer i appears in the diagram.*

Example 2. A possible tableau for $\lambda = (3, 2, 2, 1)$ with filling $\mu = (2, 4, 2)$ is

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 2 & 2 \\ \cline{1-2} 3 & 2 \\ \cline{1-2} 1 \\ \cline{1-1}\end{array}$$

Notice that in order for the diagram to be completely filled, it is necessary for $|\lambda| = |\mu|$.

While it is possible to fill tableaux arbitrarily in this manner, the tableaux will be more useful if we impose restrictions on the filling μ. These restrictions lead us to the definition of Young tableaux.

Definition 5. *Suppose $\lambda \vdash n$. A **Young tableau of shape** λ, is an array T obtained by filling the Ferrers diagram of shape λ so that the filling is weakly increasing across each row and strictly increasing down each column. A Young tableau of shape λ is also called a λ-tableau. A tableau T is **standard** if the rows and columns of T are increasing sequences. That is, T is filled with the numbers $1, 2, \ldots, n$ bijectively. A tableau T is **semistandard** if the filling is weakly increasing across each row and strictly increasing down each column.*

Example 3. Given $\lambda = (3, 2, 2, 1)$ as above, a standard tableau T would be

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 \\ \cline{1-2} 6 & 7 \\ \cline{1-2} 8 \\ \cline{1-1}\end{array}$$

Suppose we relax the necessary conditions of a standard tableau by allowing repetition, which leads to a semistandard tableau. Then a possible semistandard tableau is

$$\begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline 2 & 2 \\ \cline{1-2} 3 & 4 \\ \cline{1-2} 4 \\ \cline{1-1}\end{array}$$

Notice that 1 and 4 appear twice, and 2 appears three times. The filling in this example is $\mu = (2, 3, 1, 2)$.

Recall the definition of a filling $\mu = (\mu_1, \mu_2, \dots, \mu_k)$. In order for a tableau to be semistandard, the number of boxes in the first column must be less than k. For example, given $\mu = (2, 3)$, it is not possible to have a corresponding semistandard tableau of shape $\lambda = (3, 1, 1)$. Although $|\mu| = |\lambda|$, it is not possible to fill the three boxes of the first column in a strictly increasing manner with the given μ.

This paper will focus on semistandard tableaux since this type of tableaux gives rise to Schur polynomials and also appears in the Littlewood-Richardson rule.

Now we are ready to define the Schur polynomial determined by a partition λ and the corresponding Young tableaux.

Definition 6. *Fix λ and a bound N on the size of the entries in each semistandard tableau T. Let $\boldsymbol{x}^T = \Pi_{i=1}^{N} x_i^j$, where j = the number of i's in T. Then the* ***Schur polynomial*** *is* $s_\lambda(x_1, \dots x_N) := \displaystyle\sum_{semistandard\ T} \boldsymbol{x}^T$.

Example 4. Let $\lambda = (2, 1)$. Then the list of possible semistandard tableaux of shape λ where $N = 3$ are

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 3 \\ \cline{1-1}\end{array}$$

The corresponding Schur polynomial is given by

$$s_{(2,1)}(x_1, x_2, x_3) = x_1^2 x_2 + x_1^2 x_3 + x_1 x_2^2 + x_1 x_2 x_3 + x_1 x_3 x_2 + x_1 x_3^2 + x_2^2 x_3 + x_2 x_3^2$$

The Schur polynomial $s_{(2,1)}$ is a homogeneous symmetric polynomial of degree $|\lambda| = 3$. Notice that each term corresponds to one possible tableau, and the degree of each term is three which corresponds to the number of boxes in the given tableau, i.e., $|\lambda|$. To understand homogeneous symmetric polynomials, we now introduce symmetric functions.

2.2 Introduction to Symmetric Functions

Schur functions, as special symmetric polynomials, have been introduced. There are other types of symmetric functions as well, among them elementary symmetric functions and power sum symmetric functions. Each of these families of symmetric functions forms a basis for the vector space of symmetric functions.

Definition 7. *A polynomial $f \in \mathbb{C}[x_1, x_2, \ldots, x_n]$ is a **symmetric polynomial** if it is invariant under any permutation σ of the subscripts $1, 2, \ldots, n$. That is,*

$$f(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)}) = f(x_1, x_2, \ldots, x_n)$$

for any $\sigma \in \mathcal{S}_n$.

Symmetric polynomials are a subring of the ring of multivariate polynomials. Let the ring of symmetric polynomials be denoted $\mathbb{C}[x_1, x_2, \ldots, x_n]^{\mathcal{S}_n}$.

We also say that a monomial $x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_n^{\lambda_n}$ has degree d where $d = \sum_i \lambda_i$. Then $f(x_1, \ldots, x_n) \in \mathbb{C}[x_1, \ldots, x_n]$ is **homogeneous of degree d** if every monomial of f has degree d. We will find that elementary symmetric functions, power sum symmetric functions, and Schur polynomials are homogeneous of some degree.

Example 5. Consider $f = x_1x_2x_3 - 2x_1x_2 - 2x_1x_3 - 2x_2x_3$ which is a symmetric polynomial since it is invariant under every permutation $\pi \in S_3$. Then consider $g = x_1 + x_2x_3$ which is not a symmetric polynomial. Under permutation (12), the polynomial g becomes $g' = x_2 + x_1x_3$ which is not equal to g.

Definition 8. *Given variables $x_1, x_2, \ldots, x_n$, we define the **elementary symmetric polynomials**, $\sigma_1, \sigma_2, \ldots, \sigma_n \in k[x_1, x_2, \ldots, x_n]$, as*

$$\begin{aligned}
\sigma_1 &= x_1 + x_2 + \ldots + x_n \\
\sigma_2 &= x_1x_2 + x_1x_3 + \ldots + x_2x_3 + \ldots + x_{n-1}x_n \\
\sigma_3 &= x_1x_2x_3 + x_1x_2x_4 + \ldots + x_2x_3x_4 + \ldots + x_{n-2}x_{n-1}x_n \\
&\vdots \\
\sigma_r &= \sum_{i_1 < i_2 < \cdots < i_r} x_{i_1} x_{i_2} \cdots x_{i_r} \\
&\vdots \\
\sigma_n &= x_1x_2 \cdots x_n,
\end{aligned}$$

where k is a field.

Every symmetric polynomial can be written uniquely in terms of elementary symmetric polynomials. Notice that each σ_r is the sum of all monomials that are products of r distinct variables, and every term of σ_r has total degree r.

Definition 9. *The nth **power sum symmetric function** is $p_n = \sum_{i\geq 1} x_i^n$.*

Example 6. Given x_1, x_2, x_3, we find that

$$\sigma_1 = x_1 + x_2 + x_3, \qquad \sigma_2 = x_1x_2 + x_1x_3 + x_2x_3, \qquad \sigma_3 = x_1x_2x_3$$

and

$$p_1 = x_1 + x_2 + x_3, \qquad p_2 = x_1^2 + x_2^2 + x_3^2, \qquad p_3 = x_1^3 + x_2^3 + x_3^3.$$

Notice that each of the σ_r and p_r are symmetric polynomials, that is, applying any permutation $\pi \in \mathcal{S}_n$ to the subscripts of the x_i will not change the polynomial. This is true for any symmetric polynomial in n indeterminates. Notice that the σ_r and p_r are also homogeneous of degree r.

2.3 Symmetric Polynomials to Schur Polynomials

Since the Schur polynomials form a basis for the space of symmetric polynomials, it is possible to express any symmetric polynomial in terms of Schur polynomials. There are many algorithms which accomplish this. The following algorithm, proposition, and corresponding definitions come from [4]. To follow this algorithm, we need to introduce the alternating group.

Definition 10. *Let A_n denote the alternating group, the subgroup of $\mathcal{S}_n$ consisting of all even permutations.*

Let $\mathbb{C}[x_1, \ldots, x_n]^{A_n}$ denote the subring of polynomials fixed by all even permutations, that is, $\mathbb{C}[x_1, \ldots, x_n]^{\mathcal{S}_n} \subseteq \mathbb{C}[x_1, \ldots, x_n]^{A_n}$. Since the polynomial

$$D(x_1, \ldots, x_n) = \prod_{1\leq i<j\leq n} (x_i - x_j)$$

is fixed by all even permutations and not fixed by any odd permutation, we find that $\mathbb{C}[x]^{\mathcal{S}_n} \subseteq \mathbb{C}[x]^{A_n}$. D will be used in the upcoming algorithm and also happens to be an example of an antisymmetric polynomial which we now define.

Definition 11. *Each polynomial $g \in \mathbb{C}[x_1, \ldots, x_n]^{A_n}$ is called an **antisymmetric** polynomial meaning*

$$h(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)}) = \operatorname{sign}(\sigma) \cdot h(x_1, x_2, \ldots, x_n)$$

for all $\sigma \in \mathcal{S}_n$.

In words, any antisymmetric polynomial is invariant under all even permutations and not fixed by any odd permutations of $\mathcal{S}_n$. Moreover, these polynomials change sign when acted upon by an odd permutation.

Now we show that any antisymmetric polynomial is divisible by D, where

$$D = \prod_{1 \leq i < j \leq n} (x_i - x_j),$$

and the result is a symmetric polynomial.

Proposition 1. *Every polynomial $f \in \mathbb{C}[x_1, \ldots, x_n]^{A_n}$ can be written uniquely in the form $f = g + hD$ where g and h are symmetric polynomials.*

Proof. Let

$$g(x_1, x_2, \ldots, x_n) := \frac{1}{2}[f(x_1, x_2, \ldots, x_n] + f(x_2, x_1, \ldots, x_n]$$

and

$$\overline{h}(x_1, x_2, \ldots, x_n) := \frac{1}{2}[f(x_1, x_2, \ldots, x_n) - f(x_2, x_1, \ldots, x_n)],$$

where g is a symmetric polynomial and $\overline{h}$ is an antisymmetric polynomial for all $\sigma \in \mathcal{S}_n$. That is,

$$\overline{h}(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)}) = \text{sign}(\sigma) \cdot \overline{h}(x_1, x_2, \ldots, x_n).$$

Since $\overline{h}$ is antisymmetric, $\overline{h}$ vanishes when one variable, say x_i, is replaced with a different variable x_j. Then $x_i - x_j$ divides $\overline{h}$ for all $1 \leq i < j \leq n$. Since $x_i - x_j$ divides, $\overline{h}$, D divides $\overline{h}$.

To show uniqueness, suppose $f = g + hD = g' + h'D$. Let $\pi \in \mathcal{S}_n$ be an odd permutation. Applying π to f, $f \circ \pi = g - hD = g' - h'D$. Add these two equations together results in

$$(g + hD - g' - h'D) + (g - hD - g' + h'D) = 2g - 2g' \Rightarrow g = g'$$

and hence $h' = h$. Therefore every polynomial f that can be fixed by even permutations can be expressed uniquely in the form $f = g + hD$ where g and h are symmetric polynomials. $\square$

Then we let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ be a partition of positive integer d with $\lambda_1 \geq \lambda_2 \geq \ldots \lambda_n \geq 1$. Associated with this partition λ is the homogeneous polynomial a_λ given by

$$a_\lambda(x_1,\dots,x_n) = \det \begin{bmatrix} x_1^{\lambda_1+n-1} & x_2^{\lambda_1+n-1} & \dots & x_n^{\lambda_1+n-1} \\ x_1^{\lambda_2+n-2} & x_2^{\lambda_2+n-2} & \dots & x_n^{\lambda_2+n-2} \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{\lambda_l} & x_2^{\lambda_l} & \dots & x_n^{\lambda_n} \end{bmatrix}$$

The degree of the polynomial $a_\lambda = d + \binom{n}{2}$. The polynomials a_λ form a basis for the vector space of all antisymmetric polynomials, and hence we can divide a_λ by D and the result is a symmetric polynomial by the previous proposition. More specifically we arrive at the definition of the Schur polynomial corresponding to the partition λ given by

$$s_\lambda = \frac{a_\lambda}{D}, \tag{1}$$

where s_λ is a symmetric polynomial that is homogeneous of degree $d = |\lambda|$.

Algorithm 1 Schur polynomials [Sturmfels]

Require: Given a homogeneous symmetric polynomial f of degree d

Ensure: Construct the unique representation of $f = \Sigma_{\lambda \vdash d} c_\lambda s_\lambda$ in terms of Schur polynomials, the s_λ.

1: If $f = 0$ then output the zero polynomial.
2: $D(x_1, x_2, \dots, x_n) := \prod_{1 \le i < j \le n} (x_i - x_j)$
3: **while** $f \neq 0$ **do**
4: Let $t_1^{v_1} t_2^{v_2} \dots t_n^{v_n}$ be the lexicographically leading monomial of f of degree d.
5: Let $\lambda = (v_1, v_2, \dots, v_n) \vdash d$ be the partition corresponding to $t_1^{v_1} t_2^{v_2} \dots t_n^{v_n}$ in f.
6: Compute A_λ using λ and n from above in

$$a_\lambda(x_1,\dots,x_n) := \det \begin{bmatrix} x_1^{\lambda_1+n-1} & x_2^{\lambda_1+n-1} & \dots & x_n^{\lambda_1+n-1} \\ x_1^{\lambda_2+n-2} & x_2^{\lambda_2+n-2} & \dots & x_n^{\lambda_2+n-2} \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{\lambda_n} & x_2^{\lambda_n} & \dots & x_n^{\lambda_n} \end{bmatrix}$$

7: $s_\lambda(x_1, x_2, \dots, x_n) := \dfrac{a_\lambda(x_1, x_2, \dots, x_n)}{D}$
8: Let c be the coefficient of $t_1^{v_1} t_2^{v_2} \dots t_n^{v_n}$ in f.
9: $f := f - c \cdot s_\lambda$
10: **end while**
11: **return** f

The definition (1) of Schur polynomials is used to convert symmetric polynomials to Schur polynomials in the above algorithm given by B. Sturmfels. [4] For those interested, there are other more efficient algorithms to accomplish this task, including the "SF" Maple package by J. Stembridge at the University of Michigan, Ann Arbor. [3] The Maple package "SP" by R. Ablamowicz [1] is helpful in analyzing and verifying symmetric polynomials, as we will see in the following example.

Example 7. Here is an example computed in Maple to illustrate Sturmfels' algorithm.

```
> f := x[1]^2+x[2]^2+x[3]^2;
```

$$f := {x_1}^2 + {x_2}^2 + {x_3}^2$$

$$A := \begin{bmatrix} {x_1}^4 & {x_2}^4 & {x_3}^4 \\ x_1 & x_2 & x_3 \\ 1 & 1 & 1 \end{bmatrix}$$

$$a := {x_1}^4\, x_2 - {x_1}^4\, x_3 - x_1\, {x_2}^4 + x_1\, {x_3}^4 + {x_2}^4\, x_3 - {x_3}^4\, x_2$$

Note that the leading monomial of f is x_1^2. Thus $n = 3$, $\lambda = (2, 0, 0)$ and $c = 0$.

```
> A := matrix(3, 3, [x[1]^(2+3-1), x[2]^(2+3-1),
> x[3]^(2+3-1), x[1]^(0+3-2), x[2]^(0+3-2), x[3]^(0+3-2),
> x[1]^0, x[2]^0, x[3]^0]);
> a := det(A);
> Dee := (x[1]-x[2])*(x[1]-x[3])*(x[2]-x[3]);
> s1 := factor(a)/Dee;
```

$$Dee := (x_1 - x_2)\,(x_1 - x_3)\,(x_2 - x_3)$$

$$a1 := (x_1 - x_3)\,(x_2 - x_3)\,(x_1 - x_2)\,({x_2}^2 + x_3\, x_2 + x_1\, x_2 + {x_3}^2 + x_1\, x_3 + {x_1}^2)$$

$$s1 := {x_2}^2 + x_3\, x_2 + x_1\, x_2 + {x_3}^2 + x_1\, x_3 + {x_1}^2$$

```
> c1 := 1; f1 := f-c1*s1;
```

$$c1 := 1$$

$$f1 := -x_3\, x_2 - x_1\, x_2 - x_1\, x_3$$

Note that the leading monomial of $f1$ is $-x_1 x_2$. Thus $n = 3$ and $\lambda = (1, 1, 0)$.

```
> A2 := matrix(3, 3, [x[1]^(1+3-1), x[2]^(1+3-1),
> x[3]^(1+3-1), x[1]^(1+3-2), x[2]^(1+3-2), x[3]^(1+3-2),
> x[1]^0, x[2]^0, x[3]^0]);
> a2 := det(A2);
```

$$A2 := \begin{bmatrix} {x_1}^3 & {x_2}^3 & {x_3}^3 \\ {x_1}^2 & {x_2}^2 & {x_3}^2 \\ 1 & 1 & 1 \end{bmatrix}$$

$$a2 := {x_1}^3\, {x_2}^2 - {x_1}^3\, {x_3}^2 - {x_1}^2\, {x_2}^3 + {x_1}^2\, {x_3}^3 + {x_2}^3\, {x_3}^2 - {x_3}^3\, {x_2}^2$$

```
>   s2 := factor(a2)/Dee;
```

$$s2 := x_3\,x_2 + x_1\,x_2 + x_1\,x_3$$

```
>   c2:=-1:  f2 := f1-c2*s2;
```

$$f2 := 0$$

```
>   evalb(f = c1*s1+c2*s2);
```

$$true$$

Note that f written in terms of Schur polynomials is $f = s1 - s2$ where $s1$ and $s2$ are given below.

```
>   s1; s2;
```

$$x_2{}^2 + x_3\,x_2 + x_1\,x_2 + x_3{}^2 + x_1\,x_3 + x_1{}^2$$
$$x_3\,x_2 + x_1\,x_2 + x_1\,x_3$$

3 $\mathcal{S}_n$ and Conjugacy Classes

Recall the symmetric group $\mathcal{S}_n$ and cycle notation. It turns out that the cycle notation used to describe permutations of $\mathcal{S}_n$ can be referenced by cycle type and also be in correspondence with a partition λ of the positive integer n. This correspondence is how Schur polynomials can be related to the representations of $\mathcal{S}_n$.

Definition 12. *If $\pi \in \mathcal{S}_n$ is the product of disjoint cycles of lengths $n_1, n_2, \ldots, n_r$, then the integers $n_1, n_2, \ldots, n_r$ are the **cycle type** of π.*

Definition 13. *In any group G, the elements g and h are **conjugates** if*

$$g = khk^{-1}$$

*for some $k \in G$. The set of all elements conjugate to a given g is called the **conjugacy class of g**. Conjugacy classes partition the group G.*

Let λ correspond to the cycle type of a permutation $\sigma \in \mathcal{S}_n$. Then we see a one-to-one correspondence between partitions of n and the conjugacy classes of $\mathcal{S}_n$.

Example 8. For example, consider $\mathcal{S}_4$. Below are the partitions of 4 and the corresponding conjugacy classes of $\mathcal{S}_4$.

$$\lambda_1 = (1,1,1,1) \to \{(e)\}$$
$$\lambda_2 = (2,1,1) \to \{(12),(13),(14),(23),(24),(34)\}$$
$$\lambda_3 = (2,2) \to \{(12)(34),(13)(24),(14)(23)\}$$
$$\lambda_4 = (3,1) \to \{(123),(132),(124),(142),(134),(143),(234),(243)\}$$
$$\lambda_5 = (4) \to \{(1234),(1432),(1423),(1324),(1342),(1243)\}$$

Notice there are five partitions of 4 that correspond to the five conjugacy classes of $\mathcal{S}_4$.

Since each Schur polynomial s_λ corresponds to the partition λ, the s_λ corresponds to the conjugacy class of $\mathcal{S}_n$ described by λ.

4 Representations

In this section we define representations and describe what it means for a representation to be irreducible. There are different ways to represent a group, but here we will introduce a representation of a group as a group of matrices. Matrix representations are easily manipulated, and lend themselves well to the concept of modules which will be presented in this section. Representations of a group G are commonly discussed in terms of G-modules.

While matrices are easy to manipulate, a group of matrices can be cumbersome to deal with. In this section we introduce characters which are the traces of the matrices associated with a representation. Characters are useful in analyzing representations. Also in this section we introduce a special relationship between representations of $\mathcal{S}_n$ and the representations of its subgroups which will be important later in our discussion of the Littlewood-Richardson rule.

4.1 Definitions

In this section we introduce definitions and propositions regarding representations and irreducibility in order to understand what it means to decompose a representation into its irreducible components.

Definition 14. *Let Mat_d be the set of all $d \times d$ matrices with entries in $\mathbb{C}$. Let the* ***complex general linear group of degree d****,denoted GL_d, be the group of all invertible matrices $X = (x_{i,j})_{d\times d}$. Then a* ***matrix representation of a group G*** *is a group homomorphism*

$$X : G \to GL_d.$$

Equivalently, to each $g \in G$ is assigned a matrix $X(g)$ such that

1. *$X(e) = I$ (the identity matrix), and*

2. *$X(gh) = X(g)X(h)$ for all $g, h \in G$.*

The parameter d is called the ***degree*** *or* ***dimension*** *of the representation.*

By mapping every element of a group G to the identity matrix, we find that every group has a trivial representation. Recall Cayley's theorem which states that every finite group is isomorphic to a permutation group. It can be shown that every finite group has a permutation matrix representation. Since we are focusing on the symmetric group, we will assume that all groups mentioned from here on are finite and hence have a matrix representation. Let us define what it means for a representation to be irreducible. To do this we describe the concept of a module.

Definition 15. *Let V be a vector space and G be a group. Let general linear group of V, denoted $GL(V)$, be the set of all invertible linear transformations of V to itself. If dim $V = d$, then $GL(V)$ and GL_d are isomorphic as groups. We say V is a* ***G-module*** *if there is a group homomorphism*

$$\rho : G \to GL(V).$$

Definition 16. *Let V be a G-module. A* ***submodule of*** *V is a subspace W that is closed under the action of G, that is.*

$$\boldsymbol{w} \in W \Rightarrow g\boldsymbol{w} \in W \text{ for all } g \in G.$$

We also say that W is a G-invariant subspace, and W is a G-module in its own right.

Now we can define what it means for a representation to be irreducible.

Definition 17. *A non-zero G-module V is* ***reducible*** *if it contains a non-trivial submodule W. Otherwise V is said to be* ***irreducible****.*

Theorem 1 (Maschke's Theorem)**.** *Let G be a finite group and let V be a nonzero G-module. Then*

$$V = W^{(1)} \oplus W^{(2)} \oplus \cdots \oplus W^{(k)},$$

where each $W^{(i)}$ is an irreducible G-submodule of V.

The proof of Maschke's Theorem can be found in [2] and follows from induction on the dimension of V and construction of the complement of submodule W.

Definition 18. *A representation of a group G is* ***completely reducible*** *if the non-zero G-module can be written as a direct sum of irreducible components.*

With this definition, Maschke's theorem could be restated to say that every representation of a finite group having positive dimension is completely reducible. So the symmetric group $\mathcal{S}_n$, as well as every subgroup of $\mathcal{S}_n$, is completely reducible.

It is also the case that the number of irreducible representations of a finite group correspond to the number of conjugacy classes of the group.

Proposition 2. *Let G, V, and W be as described in Maschke's Theorem above, where*

$$V = W^{(1)} \oplus W^{(2)} \oplus \cdots \oplus W^{(k)}.$$

Then the number of pairwise inequivalent irreducible $W^{(i)}$ equals the number of conjugacy classes of G.

Thus the number of conjugacy classes of $\mathcal{S}_n$ gives the exact number of irreducible representations of $\mathcal{S}$. In a previous Example 8, the five conjugacy classes of $\mathcal{S}_4$ were presented, along with each corresponding λ. By Maschke's Theorem and corresponding propositions, $\mathcal{S}_4$ has five conjugacy classes and five irreducible pairwise inequivalent representations.

4.2 Group Characters

Much of the information conveyed by the representation of a group can be condensed into one statistic: the traces of the matrices in the matrix representation. We will introduce group characters and discuss a few results that will be useful in proving future results about representations.

Definition 19. *Let $X(g)$, $g \in G$, be a matrix representation of the group G. Then the* ***character of*** *X is*

$$\chi(g) = \operatorname{tr} X(g),$$

where tr *denotes the trace of a matrix. If V is a G-module, then the character of V is the character of a matrix representation X corresponding to V.*

Proposition 3. *Let X be a matrix representation of a group G of degree d with character χ.*

(a) $\chi(e) = d$.

(b) If K is a conjugacy class of G, then

$$g, h \in K \Rightarrow \chi(g) = \chi(h).$$

(c) If Y is a representation of G with character ψ, then

$$X \cong Y \Rightarrow \chi(g) = \psi(g) \textit{ for all } g \in G.$$

Proof. (a) Since $X(e) = I_d$, $\chi(e) = \operatorname{tr} I_d = d$.

(b) If K is a conjugacy class of G, then $g = khk^{-1}$, so

$$\chi(g) = \operatorname{tr} X(g) = \operatorname{tr} X(k)X(h)X(k)^{-1} = \operatorname{tr} X(h) = \chi(h).$$

(c) If X and Y are both representations of G, then $Y = TXT^{-1}$ for some fixed matrix T. Since the trace is invariant under conjugation, for all $g \in G$,

$$\operatorname{tr} Y(g) = \operatorname{tr} TX(g)T^{-1} = \operatorname{tr} X(g).$$

Definition 20. *Let χ and ψ be any two functions from a group G to the complex numbers $\mathbb{C}$. The* ***inner product*** *of characters χ and ψ is*

$$\langle \chi, \psi \rangle = \frac{1}{|G|} \sum_{g \in G} \chi(g)\overline{\psi(g)}.$$

This definition can be put into another form that is useful.

Proposition 4. *Let χ and ψ be characters. Then*

$$\langle \chi, \psi \rangle = \frac{1}{|G|} \sum_{g \in G} \chi(g)\psi(g^{-1}).$$

The following proposition relates representations to characters, allowing us to compare representations by comparing their corresponding characters.

Proposition 5. *Let X be a matrix representation of G with character χ. Suppose*

$$X \cong m_1 X^{(1)} \oplus m_2 X^{(2)} \oplus \ldots \oplus m_k X^{(k)},$$

where the $X^{(i)}$ are pairwise inequivalent irreducibles with characters $\chi^{(i)}$.

1. $\langle \chi, \chi \rangle = m_1^2 + m_2^2 + \ldots + m_k^2$.
2. *X is irreducible if and only if $\langle \chi, \chi \rangle = 1$.*
3. *Let Y be another matrix representation of G with character ψ. Then*

$$X \cong Y \text{ if and only if } \chi(g) = \psi(g) \text{ for all } g \in G.$$

These properties of characters will be used to prove an important theorem and proposition in describing representations. First we give some background on representation theory of the symmetric group.

4.3 Induced Representations of the Symmetric Group

We want to construct irreducible representations of the symmetric group. We have already found that the number of irreducible representations of $\mathcal{S}_n$ is equal to the number of conjugacy classes of the group. In the case of $\mathcal{S}_n$, the number of conjugacy classes is equal to the number of partitions of n. So we have a one-to-one correspondence between irreducible representations of $\mathcal{S}_n$ and partitions of n.

Given a group G and a subgroup H, it is possible to find representations of G from the representations of H and vice versa. In constructing the representations of $\mathcal{S}_n$, we want to consider inducing a representation from a subgroup of $\mathcal{S}_n$ to the entire group $\mathcal{S}_n$.

Recall the one-to-one correspondence between irreducible representations of $\mathcal{S}_n$ and partitions of n mentioned earlier. To see this correspondence, let

$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l) \vdash n$$

There is a subgroup $\mathcal{S}_\lambda \in \mathcal{S}_n$ corresponding to λ that is an isomorphic copy of

$$\mathcal{S}_{\lambda_1} \times \mathcal{S}_{\lambda_2} \times \ldots \times \mathcal{S}_{\lambda_l}.$$

In general, $\mathcal{S}_\lambda = \mathcal{S}_{(\lambda_1,\lambda_2,\ldots,\lambda_l)}$ and $\mathcal{S}_{\lambda_1} \times \mathcal{S}_{\lambda_2} \times \cdots \times \mathcal{S}_{\lambda_l}$ are isomorphic as groups, and $\mathcal{S}_{\lambda_1} \times \mathcal{S}_{\lambda_2} \times \cdots \times \mathcal{S}_{\lambda_l}$ is called the **Young subgroup** of $\mathcal{S}_n$.

Definition 21. *Let G and H be groups with $H \leq G$ and fix a transversal $t_1, \ldots, t_l$ for the left cosets of H, i.e., $G = t_1 H \biguplus \cdots \biguplus t_l H$, where $\biguplus$ denotes disjoint union. If Y is a representation of H, then the corresponding* ***induced representation*** *$Y\uparrow_H^G$ assigns to each $g \in G$ the block matrix*

$$Y\uparrow_H^G (g) = (Y(t_i^{-1} g t_j)) = \begin{pmatrix} Y(t_1^{-1}gt_1) & Y(t_1^{-1}gt_2) & \ldots & Y(t_1^{-1}gt_l) \\ Y(t_2^{-1}gt_1) & Y(t_2^{-1}gt_2) & \ldots & Y(t_2^{-1}gt_l) \\ \vdots & \vdots & \ddots & \vdots \\ Y(t_l^{-1}gt_1) & Y(t_l^{-1}gt_2) & \ldots & Y(t_l^{-1}gt_l) \end{pmatrix}$$

where $Y(g)$ is the zero matrix if $g \notin H$.

The following theorems, propositions and proofs are borrowed from [2].

Theorem 2. *Suppose $H \leq G$ has transversal $\{t_1, t_2, \ldots, t_l\}$ and let Y be a matrix representation of H. Then $X = Y\uparrow_H^G$ is a representation of G.*

Proof. We will prove that $X(g)$ is always a block permutation matrix, i.e., every row and column contains exactly one nonzero block $Y(t_i^{-1}gt_j)$. Without loss of generality, consider the first column of $X(g)$. We want to show that there is a unique element of H in $\{t_1^{-1}gt_1, t_2^{-1}gt_1, \ldots, t_l^{-1}gt_1\}$. But $gt_1 \in t_iH$ for exactly one of the t_i in the transversal, so $t_i^{-1}gt_1 \in H$ is the unique element desired.

To show that $X(g)$ is a representation, we need to show that $X(e)$ is the identity matrix and $X(g)X(h) = X(gh)$ for all $g, h \in G$. We find that $X(e)$ is the identity matrix as a direct result of the process of inducing a representation.

Next we show that $X(g)X(h) = X(gh)$. Consider the $(i.j)$ block on both sides of this equation. It is enough to prove that

$$\sum_k Y(t_i^{-1}gt_k)Y(t_k^{-1}ht_j) = Y(t_i^{-1}ght_j).$$

To simplify notation, let $a_k = t_i^{-1}t_k, b_k = t_k^{-1}ht_j$, and $c = t_i^{-1}ght_j$. Note that $a_kb_k = c$ for all k, so we will prove that

$$\sum_k Y(a_k)Y(b_k) = Y(c).$$

Case I: If $Y(c) = 0$, then $c \notin H$, and so either $a_k \notin H$ or $b_k \notin H$ for all k. This implies either $Y(a_k)$ or $Y(b_k)$ is zero for each k, but in either case $\sum_k Y(a_k)Y(b_k) = 0 = Y(c)$.

Case II: If $Y(c) \neq 0$, then $c \in H$. Since $c \in H$, then $a_k \in H$ for some k. Let m be the unique index such that $a_m \in H$. Then $b_m = a_m^{-1}c \in H$, and so

$$\sum_k Y(a_k)Y(b_k) = Y(a_m)Y(b_m) = Y(a_mb_m) = Y(c).$$

It is important to note that the induced representation is not guaranteed to be irreducible, regardless of the irreducibility of the original representation. The induced representation is independent of the choice of transversal, it is dependent only on the subgroup, as seen in the following proposition.

Proposition 6. *Consider $H \leq G$ and a matrix representation Y of H. Let $\{t_1, t_2, \ldots, t_l\}$ and $\{s_1, s_2, \ldots, s_l\}$ be two transversals for H giving rise to representation matrices X and Z, respectively, for $Y\uparrow_H^G$. Then X and Z are equivalent.*

Proof. Let χ, ψ, and ϕ be the characters of X, Y, and Z, respectively. Then it suffices to show that $\chi = \phi$ by Proposition 5. We have that

$$\chi(g) = \sum_i \operatorname{tr} Y(t_i^{-1}gt_i) = \sum_i \psi(t_i^{-1}gt_i),$$

where $\psi(g) = 0$ if $g \notin H$. Similarly,

$$\phi(g) = \sum_i \psi(s_i^{-1} g s_i).$$

Since t_i and s_i are both traversals, we can permute subscripts if necessary and obtain $t_i H = s_i H$ for all i. Now $t_i = s_i h_i$, where $h_i \in H$ for all i, so $t_i^{-1} g t_i = h_i s_i^{-1} g s_i h_i$. This implies that $t_i^{-1} g t_i \in H$ if and only if $s_i^{-1} g s_i \in H$, and when both are in H, they are in the same conjugacy class. Since characters are constant on conjugacy classes of H, it follows that $\psi(t_i^{-1} g t_i) = \psi(s_i^{-1} g s_i)$. Then we have

$$\chi(g) = \sum_i \psi(t_i^{-1} g t_i) = \sum_i \psi(s_i^{-1} g s_i) = \phi(g).$$

As mentioned before the induced representation is not necessarily irreducible, so we need a method for finding the irreducible modules of the induced representation. The irreducible modules of the symmetric group are called Specht modules and are denoted $\mathcal{S}^\lambda$. Specht modules are outside of the scope of this paper, but [2] is an excellent reference. The Littlewood-Richardson rule can be used to find the multiplicities of the irreducible Specht modules in the decomposition of the induced representation of the symmetric group.

4.4 Tensor Product Representations

Let us switch back to the main focus of this paper, the Littlewood-Richardson rule. The Littlewood-Richardson rule uses the tensor product representation, so let us now introduce tensor products and tensor product representations.

Definition 22. *Let $X = (x_{i,j})$ and Y be matrices. Then their **tensor product** is the block matrix*

$$X \otimes Y = (x_{i,j} Y) = \begin{bmatrix} x_{1,1}Y & x_{1,2}Y & \dots \\ x_{2,1}Y & x_{2,2}Y & \dots \\ \vdots & \vdots & \ddots \end{bmatrix}.$$

Definition 23. *Let G and H have matrix representations X and Y, respectively. The **tensor product representation**, $X \otimes Y$, assigns to each $(g,h) \in G \times H$ the matrix*

$$(X \otimes Y)(g,h) = X(g) \otimes Y(h)$$

Theorem 3. *Let X and Y be matrix representations for groups G and H, respectively.*

1. *Then $X \otimes Y$ is a representation of $G \times H$.*

2. If X, Y, and $X \otimes Y$ have characters denoted by χ, ψ, and $\chi \otimes \psi$, respectively, then

$$(\chi \otimes \psi)(g, h) = \chi(g)\psi(h)$$

for all $(g, h) \in G \times H$.

The following theorem is important in understanding the Littlewood-Richardson rule. As mentioned before, the rule will give us the multiplicities of the irreducible representations in the tensor product decomposition of a group. The following theorem ensures that the tensor product results in a representation, and also ensures that the tensor product decomposition really is the complete decomposition of the groups in question.

Theorem 4. *Let G and H be groups.*

(1) If X and Y are irreducible representations of G and H, respectively, then $X \otimes Y$ is an irreducible representation of $G \times H$.

(2) If $X^{(i)}$ and $Y^{(i)}$ are complete lists of inequivalent irreducible representations for G and H, respectively, then $X^{(i)} \otimes Y^{(i)}$ is a complete list of inequivalent irreducible $G \times H$-modules.

Proof. (1) Recall that if ϕ is any character, then a representation is irreducible if and only if $\langle \phi \, , \, \phi \rangle = 1$. Let X and Y have characters χ and ψ respectively. Then we have

$$\langle \chi \otimes \psi, \chi \otimes \psi \rangle = \frac{1}{|\, G \times H|} \sum_{(g,h) \in G \times H} (\chi \otimes \psi)(g, h)(\chi \otimes \psi)(g^{-1}, h^{-1})$$

By the previous theorem and the fact that $|\, G \times H| = |\, G||\, H|$, we have

$$\langle \chi \otimes \psi, \chi \otimes \psi \rangle = \frac{1}{|\, G||\, H|} \sum_{g \in G, \, h \in H} (\chi(g)\psi(h)\chi(g^{-1})\psi(h^{-1})).$$

Characters are integers, and hence χ and ψ can be moved around to give

$$\langle \chi \otimes \psi, \chi \otimes \psi \rangle = \frac{1}{|\, G||\, H|} \sum_{g \in G, \, h \in H} (\chi(g)\chi(g^{-1})\psi(h)\psi(h^{-1})).$$

Notice that $\chi(g)\chi(g^{-1})$ is independent of h and similarly $\psi(h)\psi(h^{-1})$ is independent of g. So we may rewrite our sum to get

$$\langle \chi \otimes \psi, \chi \otimes \psi \rangle = \left(\frac{1}{|\, G|} \sum_{g \in G} \chi(g)\chi(g^{-1}) \right) \left(\frac{1}{|\, H|} \sum_{h \in H} \psi(h)\psi(h^{-1}) \right).$$

Using Proposition 4, we can simplify this equation to

$$\langle \chi \otimes \psi, \chi \otimes \psi \rangle = \langle \chi, \chi \rangle \langle \psi, \psi \rangle.$$

By Proposition 5, we have that

$$\langle \chi \otimes \psi, \chi \otimes \psi \rangle = 1 \cdot 1 = 1.$$

So we have that $\langle \chi \otimes \psi, \chi \otimes \psi \rangle = 1$ and by Proposition 5, the character $\chi \otimes \psi$ is irreducible. $\square$

Proof. (2) Let $X^{(i)}$ and $Y^{(j)}$ have characters $\chi^{(i)}$ and $\psi^{(j)}$, respectively. Following the proof in part (1) we have

$$\langle \chi^{(i)} \otimes \psi^{(j)}, \chi^{(k)} \otimes \psi^{(l)} \rangle \ = \ \langle \chi^{(i)}, \chi^{(k)} \rangle \, \langle \psi^{(j)}, \psi^{(l)} \rangle \ = \ \delta_{i,k} \delta_{j,l}.$$

Then by Proposition 5 we see that $\chi^{(i)} \otimes \psi^{(j)}$ are pairwise inequivalent.

To prove that the list is complete, it is enough to show that the number of such representations is the number of conjugacy classes of $G \times H$. But the number of conjugacy classes of $G \times H$ is the number of conjugacy classes of G times the number of conjugacy classes of H, which is in turn the number of $X^{(i)} \otimes Y^{(j)}$. $\square$

Armed with knowledge of tableaux, Schur polynomials, and representations, we are ready to introduce the Littlewood-Richardson rule as our tool to find the irreducibles in a group representation.

5 Littlewood-Richardson Rule

As mentioned before, the Littlewood-Richardson rule decomposes a representation into irreducibles by looking at the product of corresponding Schur polynomials.

The product of two Schur polynomials can be decomposed into a linear combination of other Schur polynomials. Sturmfels' algorithm 1 can be used for this decomposition.

The coefficients of the resulting Schur polynomial are natural numbers called **Littlewood-Richardson coefficients**, named after the Littlewood-Richardson rule used to calculate them. These coefficients give the multiplicities of the irreducibles in the decomposition of the tensor product of irreducible representations in the general linear group.

Definition 24. *Let μ and v be arbitrary partitions. The **Littlewood-Richardson coefficients**, denoted $C^{\lambda}_{v\mu}$, are defined by*

$$s_v \cdot s_\mu = \sum_{\lambda} C^{\lambda}_{v,\mu} s_\lambda$$

where $|v| + |\mu| = |\lambda|$. These numbers count tensor product multiplicities of irreducible representations of $GL_d(\mathbb{C})$.

In the language of Specht modules, the Littlewood-Richardson coefficients are the $C^{\lambda}_{v,\mu}$, or multiplicities of Specht modules, in

$$(\mathcal{S}^{\mu} \otimes \mathcal{S}^{v})\uparrow^{S_n} = \bigoplus_{\lambda} C^{\lambda}_{v,\mu} \mathcal{S}^{\lambda}$$

where $|\mu| + |v| = n$.

The Littlewood-Richardson coefficients $C^{\lambda}_{v,\mu}$ can be determined combinatorially by considering all possible fillings μ of the skew tableau λ/v subject to conditions described in the Littlewood-Richardson rule which will be introduced shortly. In order to define the Littlewood-Richardson rule we first explore skew tableaux and row lattice permutations.

Definition 25. *Let λ and v be Ferrers diagrams such that $v \subseteq \lambda$. The corresponding **skew diagram** is the set of cells*

$$\lambda/v = \{c : c \in \lambda \text{ and } c \notin v\}.$$

Example 9. Let $\lambda = (3, 3, 2, 1)$ and $v = (2, 1, 1,)$ with filling $\mu = (1, 2, 2)$

•	•	1
•	2	2
•	3	
3		

Notice that it is necessary for $|v| + |\mu| = |\lambda|$ in order to fill the tableau, hence the condition was also included in the definition of Littlewood-Richardson coefficients.

Definition 26. *A **lattice permutation** is a sequence of positive integers $\pi = i_1\, i_2\, \ldots\, i_n$ such that, for any prefix $\pi_k = i_1\, i_2\, \ldots\, i_k$ and any positive integer l, the number of l's in π_k is at least as large as the number of $(l+1)$'s in that prefix.*

Example 10. Let $\pi = 1\,1\,2\,3\,2\,1\,3$. In this case π is a lattice permutation because 1 appears first, and 2 1's appear before 2 2's, and the 2 2's appear before 2 3's when reading from left to right.

On the contrary $\pi = 1\,2\,3\,2\,1\,1\,3$ is not a lattice permutation since 2 2's appear before 2 1's when reading from left to right.

Lattice permutations are also a method for encoding standard tableaux. Recall that standard tableaux have a bijective filling. That is, each element $1, 2, \ldots, n$ appears exactly once. Given a standard tableau T with n elements, form a sequence $\pi = i_1\, i_2\, \ldots\, i_n$ where $i_k = i$ if k appears in row i of T.

Example 11. Consider the tableau

$$\begin{array}{|c|c|c|}\hline 1 & 4 & 5 \\ \hline 2 & 6 & 7 \\ \hline 3 & 8 \\ \cline{1-2}\end{array}$$

The corresponding lattice permutation is given by $\pi = 1\,2\,3\,1\,1\,2\,2\,3$. Notice that π corresponds to 1 in the first row, 2 in the second row, 3 in the third row, 4 in the first row, 5 in the first row, 6 and 7 in the second row, and 8 in the third row.

Similarly, given a lattice permutation we can construct the corresponding unique standard tableau. For example, the lattice permutation $\pi = 1\,2\,3\,1\,2\,3\,1\,2$ corresponds to

$$\begin{array}{|c|c|c|}\hline 1 & 4 & 7 \\ \hline 2 & 5 & 8 \\ \hline 3 & 6 \\ \cline{1-2}\end{array}$$

Theorem 5 (Littlewood-Richardson Rule)**.** *The value of the coefficient $C^{\lambda}_{v,\mu}$ is equal to the number of semistandard tableaux T such that*

(1) T has shape λ/v and content μ.

(2) A lattice permutation results from listing the entries of each skew tableau across each row from right to left and from top to bottom.

Let us look at an example.

Example 12. Let $v = (2, 0)$ and $\mu = (2, 1)$, and we will consider two variables x_1 and x_2. Recall that $|\lambda| = |v| + |\,\mu|$. In this example $|v| = |(2, 0)| = 2$ and $|\,\mu| = |(2, 1)| = 3$, so $|\lambda| = 5$. We want to find the $C^{\lambda}_{v,\mu}$ in

$$s_{(2,0)} \cdot s_{(2,1)} = \sum_{\lambda} C^{\lambda}_{v,\mu} s_{\lambda}$$

The product $s_{(2,0)} \cdot s_{(2,1)}$ will result in a homogeneous symmetric polynomial of degree five. Since we are considering two variables, we need to consider all partitions of 5 into two parts. There are three such partitions: $(5, 0), (4, 1)$, and $(3, 2)$.

Thus our problem is to find the coefficients in

$$s_{(2,0)} \cdot s_{(2,1)} = C^{(5,0)}_{(2,0),(2,1)} s_{(5,0)} + C^{(4,1)}_{(2,0),(2,1)} s_{(4,1)} + C^{(3,2)}_{(2,0),(2,1)} s_{(3,2)}$$

To use the Littlewood-Richardson rule, we need to consider the skew tableaux corresponding to λ/v with all possible fillings μ where $v = (2,0)$, $\mu = (2,1)$, and λ can be $(5,0), (4,1)$, or $(3,2)$.

First let $\lambda = (5,0)$. We are considering skew tableau $\lambda/(2,0)$ with filling $\mu = (2,1)$. The possible skew tableaux are

$$A = \begin{array}{|c|c|c|c|c|}\hline \bullet & \bullet & 1 & 1 & 2 \\ \hline\end{array} \qquad B = \begin{array}{|c|c|c|c|c|}\hline \bullet & \bullet & 1 & 2 & 1 \\ \hline\end{array} \qquad C = \begin{array}{|c|c|c|c|c|}\hline \bullet & \bullet & 2 & 1 & 1 \\ \hline\end{array}$$

While the entries of A satisfy the conditions of a semistandard tableau, they do not satisfy the lattice permutation condition. B and C both violate the condition of being a semistandard tableau. Hence $C^{(5,0)}_{(2,0),(2,1)} = 0$.

Similarly, let $\lambda = (4,1)$. Then the possible skew tableaux are

$$A = \begin{array}{cccc}\hline \multicolumn{1}{|c|}{\bullet} & \multicolumn{1}{c|}{\bullet} & \multicolumn{1}{c|}{1} & \multicolumn{1}{c|}{2} \\ \hline \multicolumn{1}{|c|}{1} & & & \\ \cline{1-1}\end{array} \qquad B = \begin{array}{cccc}\hline \multicolumn{1}{|c|}{\bullet} & \multicolumn{1}{c|}{\bullet} & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{1} \\ \hline \multicolumn{1}{|c|}{1} & & & \\ \cline{1-1}\end{array} \qquad C = \begin{array}{cccc}\hline \multicolumn{1}{|c|}{\bullet} & \multicolumn{1}{c|}{\bullet} & \multicolumn{1}{c|}{1} & \multicolumn{1}{c|}{1} \\ \hline \multicolumn{1}{|c|}{2} & & & \\ \cline{1-1}\end{array}$$

Notice A is semistandard, but does not satisfy the lattice permutation condition. B is not semistandard. But C is both semistandard and satisfies the lattice permutation condition. So $C^{(4,1)}_{(2,0),(2,1)} = 1$.

Now let $\lambda = (3,2)$. The possible skew tableaux are

$$A = \begin{array}{ccc}\hline \multicolumn{1}{|c|}{\bullet} & \multicolumn{1}{c|}{\bullet} & \multicolumn{1}{c|}{1} \\ \hline \multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{2} & \\ \cline{1-2}\end{array} \qquad B = \begin{array}{ccc}\hline \multicolumn{1}{|c|}{\bullet} & \multicolumn{1}{c|}{\bullet} & \multicolumn{1}{c|}{1} \\ \hline \multicolumn{1}{|c|}{2} & \multicolumn{1}{c|}{1} & \\ \cline{1-2}\end{array} \qquad C = \begin{array}{ccc}\hline \multicolumn{1}{|c|}{\bullet} & \multicolumn{1}{c|}{\bullet} & \multicolumn{1}{c|}{2} \\ \hline \multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{1} & \\ \cline{1-2}\end{array}$$

In this case A is semistandard and satisfies the lattice permutation condition, B is not a semistandard tableau, and C does not satisfy the lattice permutation condition. Hence $C^{(3,2)}_{(2,0),(2,1)} = 1$.

So our product becomes

$$s_{(2,0)} \cdot s_{(2,1)} = s_{(4,1)} + s_{(3,2)}$$

This product can be verified by computing Schur polynomials that correspond to all possible semistandard λ-tableaux of shapes $(2,0), (2,1), (4,1)$, and $(3,2)$ in two variables.

When $\lambda = (2,0)$, the possible semistandard tableaux are

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline\end{array} \qquad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline\end{array} \qquad \begin{array}{|c|c|}\hline 2 & 2 \\ \hline\end{array}$$

with corresponding Schur polynomial being $s_{(2,0)} = x_1^2 + x_1x_2 + x_2^2$.

When $\lambda = (2,1)$, the possible semistandard tableaux are

$$\begin{array}{cc}\hline \multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{1} \\ \hline \multicolumn{1}{|c|}{2} & \\ \cline{1-1}\end{array} \qquad \begin{array}{cc}\hline \multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{2} \\ \hline \multicolumn{1}{|c|}{2} & \\ \cline{1-1}\end{array}$$

with corresponding Schur polynomial being $s_{(2,1)} = x_1^2x_2 + x_1x_2^2$.

When $\lambda = (4,1)$, the possible semistandard tableaux are

$$\begin{array}{|c|c|c|c|}\hline 1&1&1&1\\ \hline 2 \\ \cline{1-1}\end{array}\qquad \begin{array}{|c|c|c|c|}\hline 1&1&1&2\\ \hline 2 \\ \cline{1-1}\end{array}\qquad \begin{array}{|c|c|c|c|}\hline 1&1&2&2\\ \hline 2 \\ \cline{1-1}\end{array}\qquad \begin{array}{|c|c|c|c|}\hline 1&2&2&2\\ \hline 2 \\ \cline{1-1}\end{array}$$

with corresponding Schur polynomial being $s_{(4,1)} = x_1^4x_2 + x_1^3x_2^2 + x_1^2x_2^3 + x_1x_2^4$.

When $\lambda = (3,2)$, the possible semistandard tableaux are

$$\begin{array}{|c|c|c|}\hline 1&1&1\\ \hline 2&2 \\ \cline{1-2}\end{array}\qquad \begin{array}{|c|c|c|}\hline 1&1&2\\ \hline 2&2 \\ \cline{1-2}\end{array}$$

with corresponding Schur polynomial being $s_{(3,2)} = x_1^3x_2^2 + x_1^2x_2^3$.

Then

$$\begin{aligned} s_{(2,0)} \cdot s_{(2,1)} &= (x_1^2 + x_1x_2 + x_2^2)(x_1^2x_2 + x_1x_2^2) \\ &= x_1^4x_2 + 2x_1^3x_2^2 + 2x_1^2x_2^3 + x_1x_2^4 \\ &= (x_1^4x_2 + x_1^3x_2^2 + x_1^2x_2^3 + x_1x_2^4) + (x_1^3x_2^2 + x_1^2x_2^3) \\ &= s_{(4,1)} + s_{(3,2)} \end{aligned}$$

In terms of Specht modules for $v = (2,0)$ and $\mu = (2,1)$, the tensor product of the induced representation of $\mathcal{S}_5$ is given by

$$(\mathcal{S}^{(2,1)} \otimes \mathcal{S}^{(2,0)})\uparrow^{\mathcal{S}_5} = \oplus_\lambda C^\lambda_{v,u}\mathcal{S}^\lambda = \mathcal{S}^{(4,1)} \oplus \mathcal{S}^{(3,2)}$$

Here is a more involved example adapted from [5].

Example 13. Let $v = (2,1)$ and $\mu = (2,1)$ in three variables. We want to find the $C^\lambda_{v,\mu}$ in

$$s_{(2,1,0)} \cdot s_{(2,1,0)} = \sum_\lambda C^\lambda_{v,\mu} s_\lambda$$

It is convenient to denote $s_{(2,1)}$ by $s_{(2,1,0)}$ to be clear that we are considering three variables. The 0 acts as a placeholder temporarily and does not change the tableaux.

The product $s_{(2,1)} \cdot s_{(2,1)}$ will result in a homogeneous symmetric polynomial of degree 6. Since we are considering three variables, we need to consider the partitions of 6 into 3 parts. There are seven such partitions:

$$(6,0,0), \quad (5,1,0), \quad (4,2,0), \quad (4,1,1), \quad (3,3,0), \quad ([3,2,1), \quad (2,2,2).$$

Thus our problem is to find the coefficients in

$$
\begin{aligned}
s_{(2,1,0)} \cdot s_{(2,1,0)} = {} & C^{(6,0,0)}_{(2,1,0),(2,1,0)} s_{(6,0,0)} + C^{(5,1,0)}_{(2,1,0),(2,1,0)} s_{(5,1,0)} \\
& + C^{(4,2,0)}_{(2,1,0),(2,1,0)} s_{(4,2,0)} + C^{(4,1,1)}_{(2,1,0),(2,1,0)} s_{(4,1,1)} + C^{(3,3,0)}_{(2,1,0),(2,1,0)} s_{(3,3,0)} \\
& + C^{(3,2,1)}_{(2,1,0),(2,1,0)} s_{(3,2,1)} + C^{(2,2,2)}_{(2,1,0),(2,1,0)} s_{(2,2,2)}
\end{aligned}
$$

When $\lambda = (6, 0, 0)$, $v = (2, 1, 0)$ is not contained within λ, so $C^{(6,0,0)}_{(2,1,0),(2,1,0)} = 0$.

When $\lambda = (5, 1, 0)$, the possible resulting skew tableaux are

$$
A = \begin{array}{|c|c|c|c|c|} \hline \bullet & \bullet & 1 & 1 & 2 \\ \hline \bullet \\ \cline{1-1} \end{array} \qquad
B = \begin{array}{|c|c|c|c|c|} \hline \bullet & \bullet & 1 & 2 & 1 \\ \hline \bullet \\ \cline{1-1} \end{array} \qquad
C = \begin{array}{|c|c|c|c|c|} \hline \bullet & \bullet & 2 & 1 & 1 \\ \hline \bullet \\ \cline{1-1} \end{array}
$$

In this case, A is does not satisfy the lattice permutation condition, while B and C are not semistandard tableau. So $C^{(5,1,0)}_{(2,1,0),(2,1,0)} = 0$.

When $\lambda = (4, 2, 0)$, the possible resulting skew tableaux are

$$
A = \begin{array}{|c|c|c|c|} \hline \bullet & \bullet & 1 & 1 \\ \hline \bullet & 2 \\ \cline{1-2} \end{array} \qquad
B = \begin{array}{|c|c|c|c|} \hline \bullet & \bullet & 2 & 1 \\ \hline \bullet & 1 \\ \cline{1-2} \end{array} \qquad
C = \begin{array}{|c|c|c|c|} \hline \bullet & \bullet & 1 & 2 \\ \hline \bullet & 1 \\ \cline{1-2} \end{array}
$$

Then A is a semistandard tableau satisfying the lattice permutation condition with $\pi = 1\,1\,2$. B is not a semistandard tableau, and C does not satisfy the lattice permutation condition. So $C^{(4,2,0)}_{(2,1,0),(2,1,0)} = 1$.

When $\lambda = (4, 1, 1)$, the possible resulting skew tableaux are

$$
A = \begin{array}{|c|c|c|c|} \hline \bullet & \bullet & 1 & 1 \\ \hline \bullet \\ \cline{1-1} 2 \\ \cline{1-1} \end{array} \qquad
B = \begin{array}{|c|c|c|c|} \hline \bullet & \bullet & 1 & 2 \\ \hline \bullet \\ \cline{1-1} 1 \\ \cline{1-1} \end{array} \qquad
C = \begin{array}{|c|c|c|c|} \hline \bullet & \bullet & 2 & 1 \\ \hline \bullet \\ \cline{1-1} 1 \\ \cline{1-1} \end{array}
$$

Then A is a semistandard tableau satisfying the lattice permutation condition with $\pi = 1\,1\,2$. B does not satisfy the lattice permutation condition, and C is not a semistandard tableau. So $C^{(4,1,1)}_{(2,1,0),(2,1,0)} = 1$.

When $\lambda = (3, 3, 0)$, the possible resulting skew tableaux are

$$
A = \begin{array}{|c|c|c|} \hline \bullet & \bullet & 1 \\ \hline \bullet & 1 & 2 \\ \hline \end{array} \qquad
B = \begin{array}{|c|c|c|} \hline \bullet & \bullet & 2 \\ \hline \bullet & 1 & 1 \\ \hline \end{array} \qquad
C = \begin{array}{|c|c|c|} \hline \bullet & \bullet & 1 \\ \hline \bullet & 2 & 1 \\ \hline \end{array}
$$

Then A is a semistandard tableau satisfying the lattice permutation condition with $\pi = 1\,2\,1$. B and C are not semistandard tableau. So $C^{(3,3,0)}_{(2,1,0),(2,1,0)} = 1$.

When $\lambda = (3,2,1)$, the possible resulting skew tableaux are

$$A = \begin{array}{ccc} \bullet & \bullet & 1 \\ \bullet & 1 & \\ 2 & & \end{array} \qquad B = \begin{array}{ccc} \bullet & \bullet & 1 \\ \bullet & 2 & \\ 1 & & \end{array} \qquad C = \begin{array}{ccc} \bullet & \bullet & 2 \\ \bullet & 1 & \\ 1 & & \end{array}$$

Then B is not a semistandard tableaux. Both A and C are semistandard, but only A satisfies the lattice permutation condition with $\pi = 1\,1\,2$. So $C^{(3,2,1)}_{(2,1,0),(2,1,0)} = 2$.

When $\lambda = (2,2,2)$, the possible resulting skew tableaux are

$$A = \begin{array}{cc} \bullet & \bullet \\ \bullet & 1 \\ 1 & 2 \end{array} \qquad B = \begin{array}{cc} \bullet & \bullet \\ \bullet & 2 \\ 1 & 1 \end{array} \qquad C = \begin{array}{cc} \bullet & \bullet \\ \bullet & 1 \\ 2 & 1 \end{array}$$

Then A is a semistandard tableau satisfying the lattice permutation condition with $\pi = 1\,2\,1$. Both B and C are not semistandard tableaux. So $C^{(2,2,2)}_{(2,1,0),(2,1,0)} = 1$.

Putting all of this together, we find that

$$s_{(2,1,0)} \cdot s_{(2,1,0)} = s_{(4,2,0)} + s_{(4,1,1)} + s_{(3,3,0)} + 2s_{(3,2,1)} + s_{(2,2,2)}$$

In terms of Specht modules for $v = (2,1,0)$ and $\mu = (2,1,0)$, the tensor product of the induced representation of $\mathcal{S}_6$ is given by

$$(\mathcal{S}^{(2,1,0)} \otimes \mathcal{S}^{(2,1,0)})\!\uparrow^{\mathcal{S}_6} = \oplus_\lambda C^\lambda_{v,u} \mathcal{S}^\lambda = \mathcal{S}^{(4,2,0)} \oplus \mathcal{S}^{(4,1,1)} \oplus \mathcal{S}^{(3,3,0)} \oplus 2\mathcal{S}^{(3,2,1)} \oplus \mathcal{S}^{(2,2,2)}$$

This decomposition has been verified using the “SP” package for Maple [1].

6 Conclusions

One of the benefits of the Littlewood-Richardson rule is that we can find the multiplicity of a desired irreducible without having to compute the decomposition for an entire group. If we wanted to find the multiplicity of the irreducible component corresponding to $\lambda = (3,2,1)$, the only calculations necessary are those involving the skew tableaux corresponding to $\lambda/(2,1,0)$ with filling $\mu = (2,1,0)$. This is a nice time-saving method in that regard.

The Littlewood-Richardson rule is a powerful tool in uncovering the multiplicities of the irreducible representations in the decomposition of a group into irreducible modules. Although the decomposition into irreducibles in the general linear group and the symmetric group are defined differently, the Littlewood-Richardson rule can be used in either case. Schur polynomials and Young tableaux allow us to study group representations in a combinatorial fashion.

References

[1] R. Abłamowicz, B. Fauser, `SP` - *A Maple Package for Symmetric Polynomials*, `http://math.tntech.edu/rafal/`, April 2009

[2] B. Sagan, *The Symmetric Group:Representations, Combinatorial, Algorithms, and Symmetric Functions,* Springer-Verlag, New York, 2001

[3] J. Stembridge, `SF` - *A Maple Package for Symmetric Functions*, `http://www.math.lsa.umich.edu/~jrs/`, April 2009

[4] B. Sturmfels, *Algorithms in Invariant Theory,* Springer-Verlag, Wien, 1993

[5] A. Yong, *What is ... a Young Tableau?,* Notices of the AMS, volume 54, number 2, February 2007

7 Additional Reading

(1) D. Cox, J. Little, D. O'Shea, *Ideals, Varieties, and Algorithms*, Springer-Verlag, New York, 1997

(2) W. Fulton, *Young Tableaux,* Cambridge University Press, 1997