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USING PERIODICITY THEOREMS FOR COMPUTATIONS IN HIGHER DIMENSIONAL CLIFFORD ALGEBRAS

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Using Periodicity Theorems for Computations in Higher Dimensional Clifford Algebras

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Abstract We present different methods for symbolic computer algebra computations in higher dimensional (≥ 9) Clifford algebras using `CLIFFORD` and `Bigebra` packages for `Maple`®. This is achieved using graded tensor decompositions, periodicity theorems and matrix spinor representations over Clifford numbers. We show how to code the graded algebra isomorphisms and the main involutions, and we provide some benchmarks.

1 Introduction

Clifford algebras are used in several areas of mathematics, physics, and engineering. Since computing power has increased tremendously, in terms of available memory and actual processing speed, practical symbolic computations, say on a laptop finishing within minutes, are now possible for Clifford algebras over vectors spaces of dimensions higher than 8. When `CLIFFORD` was designed, way back in the early 90's [1], such computations were impossible. However, the design of `CLIFFORD`, restricting the vector space dimension to less than or equal to 9, incorporated from the beginning the idea that using mod-8 and other periodicity isomorphisms of real Clifford algebras allows nevertheless computations in higher dimensional algebras as well. Using periodicity theorems also 'groups' Clifford algebras as in a periodic table, the *spinorial chess board* [14]. Hence, implementing the periodicity in symbolic computations –as described in this paper– makes use of these intrinsic algebra features. Recent applications in engineering, for example when modeling geometric

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transformations in robotics, rely on real Clifford algebras in higher dimensions, for example, in $C\ell_{8,2}$ as demonstrated by [11], and thus enforce the need for using the periodicity in higher dimensions when computing with `CLIFFORD`.

In this note, we will describe three ways showing how `CLIFFORD` deals with (real) Clifford algebras of higher dimensions (≥ 9). One way is to use `Bigebra`, an extension of `CLIFFORD`, to utilize graded tensor products of Clifford algebras. Another method utilizes ungraded tensor products and periodicity isomorphisms without a need for introducing spinorial bases. The third method uses, on top of the periodicity theorems, a spinor representation of one factor of the tensor decomposition, hence computing in Clifford algebra valued matrix rings. For example, from $C\ell_{p+1,q+1} \simeq C\ell_{p,q} \otimes C\ell_{1,1} \simeq \mathrm{Mat}(2, C\ell_{p,q})$ we see that computing say in $C\ell_{8,2}$ –as needed in robotics [11]– can be done with the 2×2 matrices over $C\ell_{7,1}$. `CLIFFORD` supports computing with matrices having Clifford entries. Similar computations still make sense, e.g., for conformal symmetries using Vahlen matrices $\mathrm{Mat}(2, C\ell_{3,1}) \simeq C\ell_{4,2}$.

After recalling our basic notations and the periodicity theorems, we start explaining how we use `Bigebra` to compute with graded and ungraded tensor products of Clifford algebras. Then we proceed to the third matrix-based method indicated above. As this method relies on spinor representations of real Clifford algebras, one encounters not only real, but also complex, quaternionic, double real, and double quaternionic spinor representations. For the sake of simplicity and space, we will just deal with real representations of simple Clifford algebras. The periodicity theorems single out the signature cases where a graded algebra isomorphism is available onto an ungraded tensor decomposition and hence, by introducing spinor bases, matrix tensor products can be employed.

2 Tensor product decompositions and periodicity for $C\ell$-algebras

We study Clifford algebras over finite-dimensional (real) vector spaces. While these algebras can be defined by a universal property, for actual computations in a CAS (Computer Algebra System) we use generators and relations.

2.1 Basic notations, quadratic and bilinear forms

Let V be a finite dimensional real (or complex) vector space with scalar multiplication $\mathbb{R} \times V \to V :: (\lambda, v) \mapsto \lambda v$. A *quadratic form* is a map $Q : V \to \mathbb{R}$ such that $Q(\lambda v) = \lambda^2 Q(v)$, with associated polar bilinear form $B : V \times V \to \mathbb{R}$ derived from Q and defined as $2B(v,w) = Q(v+w) - Q(v) - Q(w)$ (hence $Q(v) = B(v,v)$). Q is called *non-degenerate* if $Q(v) = 0$ implies $v = 0$ ($v \in V, \forall w \in V, B(w,v) = 0$ implies $v = 0$). An isomorphism $V \simeq \mathbb{R}^n = \oplus^n \mathbb{R}$ defines a set of generators for V from the injections of the direct sum $i_k : \mathbb{R} \to \mathbb{R}^n :: 1 \mapsto e_k \simeq v_k$, that is a basis for V. Sylvester's theorem states that there exists a basis for V such that the quadratic form Q is diagonal with entries $\pm 1, 0$ in the real case (0 only when Q is degenerate; just $+1$'s in the non-degenerate complex case). Under these isomorphisms, the real quadratic space (Q, V) with a non-degenerate Q is isomorphic to a space $\mathbb{R}^{p,q} = (\mathbb{R}^n, \delta_{p,q})$ with a diagonal quadratic (polarized) form $\delta_{p,q}$ with p ones and q minus ones. (p,q) is called the *signature* of Q and $p+q = n = \dim V$. Furthermore we have generators e_k of $\mathbb{R}^{p,q}$ such that $Q(e_k) = +1$ for $1 \leq k \leq p$ and $Q(e_k) = -1$ for $p < k \leq p+q$.

2.2 Grassmann algebra, $\mathbb{Z}$- and $\mathbb{Z}_2$-gradings, main involutions

We can associate functorially to any (say finite, real) vector space V the Grassmann algebra of antisymmetric tensors, $\bigwedge : V \to \bigwedge V = \oplus_{i=0}^n \bigwedge^i V$. Using the linearly ordered basis $\{e_k\}$ of V ($e_k < e_l$ if $k < l$), we get a basis

$$\{e_I\} = \{e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_l} \mid I = (i_1, i_2, \ldots, i_l),\, i_1 < i_2 < \cdots < i_l,\, i_s \in \{1, \ldots, n\}\}$$

for the space $\bigwedge V$ of antisymmetric tensors. We can extend the order on V to the *inverse lexicographic order* on $\bigwedge V$. We associate to e_I a *degree* $|I|$ (number of generators e_k in e_I) and a *grade* (or *parity*) as $|I| \bmod 2$. The Grassmann algebra is $\mathbb{Z}$-graded w.r.t. the degree, that is $\bigwedge^i V \bigwedge^j V \subseteq \bigwedge^{i+j} V$. This can be restricted to a $\mathbb{Z}_2$-grading $\bigwedge V = \bigwedge_0 V \oplus \bigwedge_1 V$ w.r.t. the grade, hence decomposing into even and odd parts ($\bigwedge_0 V$ is a subalgebra, $\bigwedge_1 V$ is a $\bigwedge_0 V$-module). The subspaces $\bigwedge^i V$ have $\binom{n}{i}$ many basis elements and hence $\bigwedge V$ has dimension $2^n = \sum_i \binom{n}{i}$. Homogeneous elements $v_1 \wedge \cdots \wedge v_i$ of $\bigwedge^i V$ are called *extensors* (or *blades*). A general element $X \in \bigwedge V$ is an aggregate $X = \sum x_I e_I$ with real coefficients x_I. A presentation using generators and relations is given as

$$\bigwedge V := \langle e_k \mid e_k e_i = -e_i e_k,\, i,k \in \{1, \ldots, n\}\rangle. \tag{1}$$

The Grassmann algebra comes with two main involutions. The *grade involution* extends the map $- : V \to V :: v \mapsto -v$ (additive inverse in V) to $\hat{\ }$ on $\bigwedge V$. On generators this reads as $\hat{e}_I = (-1)^{|I|} e_I$ and it extends by linearity. For example, $\hat{e}_1 = -e_1$, $\hat{e}_{1,2,3} := \hat{e}_1 \wedge \hat{e}_2 \wedge \hat{e}_3 = (-1)^3 e_{1,2,3} = -e_{1,2,3}$ while $\hat{e}_{1,2} := \hat{e}_1 \wedge \hat{e}_2 = e_{1,2}$.

A second involution makes use of the opposite algebra. Let (A, m_A) with

$$m_A : A \times A \to A :: (a,b) \mapsto ab \tag{2}$$

be an algebra. The *opposite algebra* $A^{op} = (A, m_A^{op})$ is the same vector space A with the multiplication

$$m_A^{op} : A \times A \to A :: (a,b) \mapsto ba. \tag{3}$$

The opposite wedge product $\wedge^{op}$ is given as $u \wedge^{op} v = v \wedge u$ (no signs), producing the opposite Grassmann algebra $\bigwedge^{op} V$. The *reversion involution* on $\bigwedge V$ extends the identity map $1 : V \to V$ to $\bigwedge V$ in such a way that $\mathrm{rev} \circ \wedge^{op} = \wedge \circ (\mathrm{rev} \otimes \mathrm{rev})$. In other words, $\mathrm{rev} : \bigwedge V \to \bigwedge V^{op}$ is a Grassmann algebra isomorphism. As rev is invertible, this can be used to define $\wedge^{op}$. On basis elements we have $\widetilde{e}_I = (-1)^{|I|(|I|-1)/2} e_I$ with the usual notation $\mathrm{rev}(u) = \widetilde{u}$.

Let $V^* = [V, \mathbb{R}]$ be the dual vector space (linear forms). One finds that $\bigwedge V^* \simeq (\bigwedge V)^*$ is the dual Grassmann algebra, and we can define a pairing

$$\langle - \mid - \rangle : \bigwedge V^* \times \bigwedge V \to \mathbb{R} :: (e^*_I, e_K) \mapsto \begin{cases} \det(e^*_i(e_k)), & \text{when } |I| = |J|; \\ 0, & \text{when } |I| \neq |J|. \end{cases} \quad (4)$$

The *interior product* $\lrcorner$ is defined as the adjoint w.r.t. the duality pairing of multiplication in $\bigwedge V^*$, that is $\langle u \mid v \lrcorner w \rangle := \langle \tilde{v}^* \wedge u \mid w \rangle$, $v^* \in V^*$. Let $x, y \in V$ and $u, v, w \in \bigwedge V$ then $\lrcorner$ is defined operationally by the rules (Chevalley construction)

$$x \lrcorner y = \langle x \mid y \rangle, \;\; x \lrcorner (u \wedge v) = (x \lrcorner u) \wedge v + \hat{u} \wedge (x \lrcorner v), \;\; (u \wedge v) \lrcorner w = u \lrcorner (v \lrcorner w). \quad (5)$$

Using a nondegenerate bilinear form B as duality, one obtains $x \lrcorner y = B(x, y)$.

2.3 (Real) Clifford algebras

Given a quadratic space (V, Q) with a non-degenerate Q we can functorially associate to it the universal Clifford algebra $C\ell(V, Q)$ [19]. Given the isomorphism $(V, Q) \simeq \mathbb{R}^{p,q}$ this provides the Clifford algebra $C\ell(\mathbb{R}^n, \delta_{p,q})$ $(n = p + q)$ also often denoted as $\mathbb{R}^{p,q}$. As vector spaces one has $\mathbb{R}^{p,q} = \bigwedge \mathbb{R}^n$, hence we have a Grassmann basis $\{e_I\}$ spanning the vector space underlying the Clifford algebra. The Clifford product can be implemented in various ways. A Clifford algebra presentation reads

$$C\ell_{p,q} := \langle e_i \mid e_i e_j + e_j e_i = 0, i \neq j; e_i^2 = 1 \text{ if } 1 \leq i \leq p, \text{ otherwise } e_i^2 = -1 \rangle \quad (6)$$

where the Clifford multiplication is juxtaposition and $i, j \in \{1, \ldots, n\}$. Another way to define the Clifford multiplication uses the interior multiplication and the Clifford map γ. Let $x \in V$, $u \in \bigwedge V$ then $\gamma_x u = xu = x \lrcorner u + x \wedge u$ and extend by (5) with $x \lrcorner y = B(x, y)$ and linearity. Defining $C\ell \equiv C\ell(V, Q)$ as a quotient of the tensor algebra it can be seen that it is not $\mathbb{Z}$-graded, but only $\mathbb{Z}_2$-graded, $C\ell = C\ell_0 \oplus C\ell_1$ with $C\ell_0$ the even sub-Clifford algebra and $C\ell_1$ a $C\ell_0$-module. In terms of the Grassmann $\mathbb{Z}$-degrees, it is a *filtered algebra* $C\ell^i C\ell^j \subseteq \oplus_{r=|i-j|}^{i+j} C\ell^r$.[1]

Suppose the quadratic space (V, Q) with a non-degenerate Q decomposes as $(V, Q) = (V_1 + V_2, Q_1 \perp Q_2)$ with restricted quadratic forms $Q_1 = Q|_{V_1}$ on V_1 and $Q_2 = Q|_{V_2}$ on V_2. The Grassmann algebra functor is exponential, that is,

$$\bigwedge(V_1 + V_2) = \bigwedge V_1 \hat{\otimes} \bigwedge V_2 \quad (7)$$

[1] Here, we identify $C\ell^i(V) \equiv \bigwedge^i V$ as vector spaces.

(the graded tensor product $\hat{\otimes}$ is defined below and we use equality for categorical isomorphisms). Similarly we get for Clifford algebras

$$C\ell(V_1+V_2, Q_1 \perp Q_2) = C\ell(V_1, Q_1) \,\hat{\otimes}\, C\ell(V_2, Q_2), \tag{8}$$

and it is this decomposition which will be used below to compute in `CLIFFORD` in dimensions ≥ 9. For Clifford algebras with non-symmetric bilinear forms such a decomposition is in general *not* direct, see [18].

2.4 Tensor products of (graded) algebras

Let (A, m_A) and (B, m_B) be $\mathbb{K}$-algebras. That is, we have right and left scalar multiplications $\rho : A \times \mathbb{K} \to A$ and $\lambda : \mathbb{K} \times B \to B$. This allows to define the tensor product $A \otimes_{\mathbb{K}} B$ as a universal object via a co-equalizer[2] c of two arrows $\rho \circ 1$ and $1 \circ \lambda$

$$A \times \mathbb{K} \times B \underset{1\circ\lambda}{\overset{\rho\circ 1}{\rightrightarrows}} A \times B \xrightarrow{\;c\;} A \otimes_{\mathbb{K}} B \ , \tag{9}$$

which produces the common relations such as $a\lambda \otimes b = a \otimes \lambda b$, multilinearity etc. We have (vector space) injections $i_A : A \to A \otimes B$ and $i_B : B \to A \otimes B$, and want to transport the algebra structure too. We define the *product algebra* on $A \otimes B$ from the algebra structures on A and B as follows:

$$m_{A\otimes B} := (m_A \otimes m_B)(1 \otimes \sigma \otimes 1), \tag{10}$$

where $\sigma : A \otimes B \to B \otimes A :: (a,b) \mapsto (b,a)$ is the *switch* of tensor factors. On elements:

$$(a \otimes b)(a' \otimes b') = aa' \otimes bb' . \tag{11}$$

Note that we had to switch a' and b, assuming they commute. As we work in a $\mathbb{Z}_2$-graded setting $A = A_0 + A_1$, $B = B_0 + B_1$, this switch has to be replaced with a *graded switch*

$$\hat{\sigma} : A \,\hat{\otimes}\, B \to B \,\hat{\otimes}\, A :: (a,b) \mapsto (-1)^{|a||b|}(b,a), \tag{12}$$

(on homogeneous elements and extended by linearity). The graded multiplication $m_{A \,\hat{\otimes}\, B}$ is then given by

$$m_{A \,\hat{\otimes}\, B} := (m_A \,\hat{\otimes}\, m_B)(1 \,\hat{\otimes}\, \hat{\sigma} \,\hat{\otimes}\, 1). \tag{13}$$

Using this setup, the injections above become algebra isomorphisms $i_A : a \mapsto (a \,\hat{\otimes}\, 1)$ and $i_B : b \mapsto (1 \,\hat{\otimes}\, b)$.

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[2] See [22].

In the Grassmann algebra case, splitting the space $V = V_1 + V_2$ with n basis vectors e_i into two sets with, respectively, p $(1 \leq i \leq p)$ and q $(p < i \leq n)$ vectors, we get the maps $e_i \mapsto e_i \hat{\otimes} 1$ $(i \leq p)$ and $e_j \mapsto 1 \hat{\otimes} e_j$ $(p < j \leq n)$. In the CAS computations below we will *standardize* the indices, that is, we will reindex $j \mapsto j - p$ so that $i \in \{1, \ldots, p\}$ and $j - p \in \{1, \ldots n - p\}$. The graded tensor product ensures that we still have the desired anti-commutation relations

$$(e_i \hat{\otimes} 1)(1 \hat{\otimes} e_j) = (e_i \hat{\otimes} e_j) \quad \text{and} \quad (1 \hat{\otimes} e_j)(e_i \hat{\otimes} 1) = (-1)^{1 \cdot 1}(e_i \hat{\otimes} e_j)\,. \tag{14}$$

In this way we can form graded tensor products of Clifford algebras $C\ell_{p,q} \hat{\otimes} C\ell_{r,s}$ too, and that is what we aim for.

Tensor products for matrix algebras are usually called *Kronecker products*, and are taken without the grading. Let $A \simeq \mathrm{Mat}(n, \mathbb{R})$ and $B \simeq \mathrm{Mat}(m, \mathbb{R})$ be matrix algebras. Recall that matrices need a choice of basis giving matrices $[a_{ij}]$ and $[b_{ij}]$. The definition of the matrix tensor algebra $A \otimes B \simeq \mathrm{Mat}(n \cdot m, \mathbb{R})$ includes a *choice* of how to form a basis for $A \otimes B$, which consists of elements $E_{n,m} = e_{i,j} \otimes e_{k,l}$, that is, a reindexing function $[(i, j), (k, l)] \mapsto (n, m)$. One way is to define a tensor

$$[a_{i,j}] \otimes [b_{k,l}] = [a_{i,j} \cdot [b_{k,l}]] \tag{15}$$

by inserting the matrix $[b_{k,l}]$ as blocks into the matrix $[a_{i,j}]$; another obvious choice would exchange the role of $[a]$ and $[b]$. Category theory shows that the definition of the tensor algebra is unique up to a unique isomorphism depending on the particular choices. However, actual computations in a CAS need to be consistent and explicit in the choice of these isomorphisms.

2.5 Periodicity of Clifford algebras

We have seen in the previous Section that we can tensor Clifford algebras of any signature provided we employ the graded tensor product.

Theorem 1. *Let $C\ell_{p,q}$, $C\ell_{r,s}$ be two real Clifford algebras, then*

$$C\ell_{p+r,q+s} \simeq C\ell_{p,q} \hat{\otimes} C\ell_{r,s} \tag{16}$$

(which does not even need nondegeneracy, or even symmetry, of the involved forms).

This result will be used in Section 3 to describe a general method using `Bigebra` to do practical symbolic CA computations in higher dimensional real Clifford algebras of any signature.

The isomorphism in (16) is given by the procedures `bas2GTbas` (from left to right) and its inverse `GTbas2bas` (from right to left). We show both procedures in Maple worksheets posted at [7].

Using matrices over Clifford numbers, like $\mathrm{Mat}(2, C\ell_{p,q})$, needs considering *ungraded* tensors, as the matrix algebra tensor products are ungraded. Doing so employs (graded) algebra isomorphisms described on the generators of the factor Clifford algebras inside the ambient Clifford algebra. This leads to the well-known periodicity relations which are summarized in the following[3].

Theorem 2. *For real Clifford algebras we have the following periodicity theorems and isomorphisms:*

1) $C\ell_{q+1,p-1} \simeq C\ell_{p,q}$ *if* $p \geq 1$ *(see* [21]*),*
2) $C\ell_{p,q+1} \simeq C\ell_{q,p+1}$ *(see* [24]*,*
3) $C\ell_{p,q} \simeq C\ell_{p-4,q+4}$ *if* $p \geq 4$ *(see* [15,21]*),*
4) $C\ell_{p+4,q} \simeq C\ell_{p,q} \otimes C\ell_{4,0} \simeq C\ell_{p,q} \otimes \mathrm{Mat}(2,\mathbb{H})$ *(see* [24]*),*
5) $C\ell_{p,q+4} \simeq C\ell_{p,q} \otimes C\ell_{4,0} \simeq C\ell_{p,q} \otimes \mathrm{Mat}(2,\mathbb{H})$ *(see* [24]*),*
6) $C\ell_{p+1,q+1} \simeq C\ell_{p,q} \otimes C\ell_{1,1} \simeq C\ell_{p,q} \otimes \mathrm{Mat}(2,\mathbb{R}) \simeq \mathrm{Mat}(2, C\ell_{p,q})$ *(see* [21]*),*
7) $C\ell_{p,q+8} \simeq C\ell_{p,q} \otimes \mathrm{Mat}(16,\mathbb{R}) \simeq \mathrm{Mat}(16, C\ell_{p,q})$ *with* $\mathrm{Mat}(2,\mathbb{H}) \otimes \mathrm{Mat}(2,\mathbb{H}) \simeq \mathrm{Mat}(8,\mathbb{R})$ *(see* [15,21,24]*),*
8) $C\ell_{p+8,q} \simeq C\ell_{p,q} \otimes \mathrm{Mat}(16,\mathbb{R}) \simeq \mathrm{Mat}(16, C\ell_{p,q})$ *with* $\mathrm{Mat}(2,\mathbb{H}) \otimes \mathrm{Mat}(2,\mathbb{H}) \simeq \mathrm{Mat}(8,\mathbb{R})$ *(see* [15,21,24]*).*

Note that here all tensor products are *ungraded* and these structure results characterize the cases (signatures) where a *graded* isomorphism from the graded case to the ungraded case exists. The isomorphisms with matrix rings need spinor representations and will be discussed briefly in Subsection 2.6.

By way of example we show a typical isomorphism on generators for the isomorphism 6).

Example 1. Let $\{e_i\}$ be the set of orthonormal generators for $C\ell_{p,q}$ with $e_i^2 = 1$ for $i \in \{1,\ldots,p\}$ and $e_i^2 = -1$ for $i \in \{p+1,\ldots,p+q\}$, and let $f_1^2 = 1 = -f_2^2$ be the orthogonal generators for $C\ell_{1,1}$. Then, tensors $e_i \otimes f_1 f_2$, $1 \otimes f_1$, $1 \otimes f_2$ are generators for $C\ell_{p+1,q+1}$. Indeed, since $f_1 f_2 = -f_2 f_1$ and $(f_1 f_2)^2 = 1$, for $i, j \in \{1,\ldots,p+q\}$ and $i \neq j$, we find the following familiar relations in $C\ell_{p,q} \otimes C\ell_{1,1}$:

$$(e_i \otimes f_1 f_2)^2 = (e_i \otimes f_1 f_2)(e_i \otimes f_1 f_2) = e_i^2 \otimes (f_1 f_2)^2 = \begin{cases} 1 \otimes 1, & \text{if } 1 \leq i \leq p; \\ -1 \otimes 1, & \text{otherwise,} \end{cases}$$

and further:

$$\begin{gathered}
(1 \otimes f_1)^2 = 1 \otimes f_1^2 = 1 \otimes 1 = -1 \otimes f_2^2 = -(1 \otimes f_2)^2, \\
(e_i \otimes f_1 f_2)(e_j \otimes f_1 f_2) + (e_j \otimes f_1 f_2)(e_i \otimes f_1 f_2) = (e_i e_j + e_j e_i) \otimes (f_1 f_2)^2 = 0, \\
(e_i \otimes f_1 f_2)(1 \otimes f_1) + (1 \otimes f_1)(e_i \otimes f_1 f_2) = e_i \otimes (-f_1^2 f_2) + e_i \otimes (f_1^2 f_2) = 0, \\
(e_i \otimes f_1 f_2)(1 \otimes f_2) + (1 \otimes f_2)(e_i \otimes f_1 f_2) = e_i \otimes (f_1 f_2^2) + e_i \otimes (-f_1 f_2^2) = 0, \\
(1 \otimes f_1)(1 \otimes f_2) + (1 \otimes f_2)(1 \otimes f_1) = 1 \otimes (f_1 f_2 + f_2 f_1) = 0.
\end{gathered}$$

For more details see, e.g., [14,23].

[3] For additional references on the periodicity theorems see [12,13,16,20].

Theorem 3 ([14], Theorem 5.8). *With the notation as above, let V_2 have dimension $2k$ and let ω be the volume element in $C\ell(V_2,Q_2)$ with $\omega^2=\lambda\neq 0$. There exists a vector space isomorphism between the module $C\ell(V_1\oplus V_2,Q_1\perp Q_2)$ and the module $C\ell(V_1,\frac{1}{\lambda}Q_1)\otimes C\ell(V_2,Q_2)$ given on generators as $(x,y)\mapsto x\otimes\omega+1\otimes y$, and there is a graded algebra isomorphism*

$$C\ell(V_1\oplus V_2,Q_1\perp Q_2)\simeq C\ell(V_1,\frac{1}{\lambda}Q_1)\otimes C\ell(V_2,Q_2). \tag{17}$$

The involutions extend as $\widehat{(x\otimes y)}\simeq\hat{x}\otimes\hat{y}$ and $\operatorname{rev}(x\otimes y)\simeq\operatorname{rev}(x)\otimes\operatorname{rev}(y)$ if $|x|\equiv 0 \bmod 2$ even and $\operatorname{rev}(x\otimes y)\simeq\operatorname{rev}(x)\otimes\operatorname{rev}(\hat{y})$ otherwise. Then all periodicity isomorphisms in Theorem 2 are special cases of this one.[4]

To exemplify this, let (x,y) be any pair of generators with $x\in V_1$ and $y\in V_2$ which upon the embedding $V_1\oplus V_2\hookrightarrow C\ell(V_1\oplus V_2,Q_1\perp Q_2)$ we write as the sum $x+y$. Then,

$$(x+y)^2=x^2+(xy+yx)+y^2=(Q_1(x)+Q_2(y))1=(Q_1\perp Q_2)(x,y) \tag{18}$$

due to the orthogonality of x and y. On the other hand, in the (ungraded) tensor product algebra in the right-hand-side of (17) we find, as expected,

$$\begin{aligned}(x\otimes\omega+1\otimes y)^2&\\ &=(x\otimes\omega)(x\otimes\omega)+(x\otimes\omega)(1\otimes y)+(1\otimes y)(x\otimes\omega)+(1\otimes y)(1\otimes y)\\ &=x^2\otimes\omega^2+x\otimes\omega y+x\otimes y\omega+1\otimes y^2\\ &=\frac{1}{\lambda}Q_1(x)1\otimes\lambda 1+1\otimes Q_2(y)=Q_1(x)\otimes 1+1\otimes Q_2(y)\\ &=(Q_1(x)+Q_2(y))(1\otimes 1)\end{aligned} \tag{19}$$

due to the anti-commutativity $y\omega=-\omega y$ for every $y\in V_2$ assured by the even dimension of V_2 (that is ω is even). Furthermore, this last computation shows that the assumption $\omega^2=\lambda\neq 0$ and the appearance of the factor $\frac{1}{\lambda}$ modifying Q_1 in $C\ell(V_1,\frac{1}{\lambda}Q_1)$, are both necessary.

Since we are using Grassmann bases in all Clifford algebras, it is interesting to calculate the image of a Grassmann basis monomial, say of degree 2, in the product algebra on the right of (17). Let x_1,x_2 be orthogonal generators in V_1 and let y_1,y_2 be orthogonal generators in V_2. Then, the wedge product in the Clifford algebra $C\ell(V_1\oplus V_2,Q_1\perp Q_2)$ is computed as expected

$$(x_1+y_1)\wedge(x_2+y_2)=x_1\wedge x_2+x_1\wedge y_2+y_1\wedge x_2+y_1\wedge y_2. \tag{20}$$

On the other hand, since $(x_1+y_1)\wedge(x_2+y_2)=(x_1+y_1)(x_2+y_2)$, its image in $C\ell(V_1,\frac{1}{\lambda}Q_1)\otimes C\ell(V_2,Q_2)$ under the isomorphism (17) is the following rather complicated element:

[4] The requirement $\omega^2=\lambda\neq 0$ is equivalent to Q_2 being non-degenerate. See also [21, p. 218].

$$(x_1x_2)\otimes\omega^2+(x_1\wedge 1)\otimes\omega y_2+(1\wedge x_2)\otimes y_1\omega+(1\wedge 1)\otimes(y_1y_2)= \\ (x_1\wedge x_2)\otimes\lambda 1+x_1\otimes\omega y_2+x_2\otimes y_1\omega+1\otimes(y_1\wedge y_2). \quad (21)$$

The isomorphism in (17) is given by the procedures `bas2Tbas` (from left to right) and its inverse `Tbas2bas` (from right to left). In the worksheets [7] we show both procedures as well as we verify the assertions regarding the involutions.

2.6 Spinor representations, Clifford valued matrix representations

A Clifford algebra is an abstract algebra, but we may want to realize it as a concrete matrix algebra. It is, however, well known that matrix representations may be very inefficient for CAS purposes. The simplest representation is the (left) regular representation, sending $a\in A\mapsto\lambda_a=m_A(a,-)\in\mathrm{End}(A)$, the left multiplication operator by a. This representation is usually highly reducible. The smallest faithful representations of a Clifford algebra are given by spinor representations.[5] Algebraically, a spinor representation is given by a *minimal* left ideal which can be generated by left multiplication from a *primitive* idempotent $f_i=f_i^2$ with $\nexists f_k,f_l\neq 0$ idempotents such that $f_i=f_k+f_l$ and $f_kf_l=f_lf_k=0$. The vector space $S_i:=C\ell_{p,q}f_i$ is a *spinor space*, and it carries a faithful irreducible representation of $C\ell_{p,q}$ for simple algebras.[6] However, when $C\ell_{p,q}$ is not simple, and in several signatures (p,q) this space is not really a vector space, but a module over $\mathbb{K}=f_iC\ell_{p,q}f_i$ with $\mathbb{K}$ isomorphic to $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $2\mathbb{R}=\mathbb{R}\oplus\mathbb{R}$, or $2\mathbb{H}=\mathbb{H}\oplus\mathbb{H}$ depending on the signature (p,q). The spinor bi-module ${}_{C\ell}S_{\mathbb{K}}$ carries a left $C\ell_{p,q}$ and right $\mathbb{K}$ action (scalar product). Looking up tables of spinor representations [21, 24] yields that $C\ell_{p,q}$ is simple and has a real representation if $p-q\equiv 0,2 \bmod 8$, distinguished by the fact that a normalized volume element ω squares to $\omega^2=+1$ if $p-q\equiv 0,1 \bmod 4$ and $\omega^2=-1$ if $p-q\equiv 2,3 \bmod 4$. Avoiding non real $\mathbb{K}$'s, we find the isomorphisms $C\ell_{0,0}\simeq\mathbb{R}\simeq\mathrm{Mat}(1,\mathbb{R})$, $\{C\ell_{2,0},C\ell_{1,1}\}\simeq\mathrm{Mat}(2,\mathbb{R})$, $\{C\ell_{3,1},C\ell_{2,2}\}\simeq\mathrm{Mat}(4,\mathbb{R})$, $\{C\ell_{4,2},C\ell_{3,3},C\ell_{0,6}\}\simeq\mathrm{Mat}(8,\mathbb{R})$, and $\{C\ell_{8,0},C\ell_{5,3},C\ell_{4,4},C\ell_{1,7},C\ell_{0,8}\}\simeq\mathrm{Mat}(16,\mathbb{R})$, which can be looked up using the command `clidata` in `CLIFFORD`. The main reason for using these isomorphisms is that $\mathbb{R}\otimes C\ell_{p,q}\simeq C\ell_{p,q}$, so we don't have to worry about $\mathbb{K}$ tensor products of spinor spaces and use only $S\otimes_{\mathbb{R}}S^*\simeq\mathrm{Mat}(2^k,\mathbb{R})$.[7]

Example 2. A spinor basis $\{f_i^j\}$ for $C\ell_{1,1}\simeq\mathrm{Mat}(2,\mathbb{R})$ needed in the isomorphism 6) of Theorem 2, given the orthogonal generators $e_1^2=1=-e_2^2$ for $C\ell_{1,1}$ and a primitive idempotent f_i, may be chosen as

---

[5] For a complete treatment of spinor representations, see for example [2,6,8,9,14,16,17,19,21,24].

[6] For semi-simple Clifford algebras, we realize the spinor representation in $S\oplus\hat{S}$ where $\hat{S}=C\ell_{p,q}\hat{f}_i$ (see, e.g., [21]).

[7] Here, S^* denotes the dual spinor space, see e.g., [8, 9, 17, 21].

$$S_1 = C\ell_{1,1} f_1 = \mathrm{span}_{\mathbb{R}}\{f_1^1 := f_1 = \frac{1}{2}(1+e_{1,2}), f_1^2 := e_1 f_1 = \frac{1}{2}(e_1+e_2)\}. \quad (22)$$

In this basis, the following matrices represent the basis elements in $C\ell_{1,1}$:

$$[1] = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad [e_1] = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad [e_2] = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad [e_1 e_2] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (23)$$

In general there exists a transversal $\{m_i\}$ ($\{1, e_1\}$ in the previous case) which provides a basis $\{f_k^i := m_i f_k\}$ for the spinor module S_k.[8] Note that the idempotent f_1 is even, but one could choose also the idempotent $f = \frac{1}{2}(1+e_1)$ which has no definite grade.

In general, the images of the additional generators $e_i \otimes e_1 e_2$ needed to generate $C\ell_{p,q} \otimes C\ell_{1,1} \simeq \mathrm{Mat}(2, C\ell_{p,q})$ are given by $e_i \otimes [e_1 e_2]$ with entries from $C\ell_{p,q}$, and read $e_i \otimes e_1 e_2 \simeq \begin{pmatrix} e_i & 0 \\ 0 & -e_i \end{pmatrix}$. The isomorphism 6) of Theorem 2 gives, via iteration, $C\ell_{p,q} \simeq C\ell_{p-q,0} \otimes \mathrm{Mat}(2^q, \mathbb{R})$ if $p \geq q$ and $C\ell_{p,q} \simeq C\ell_{0,q-p} \otimes \mathrm{Mat}(2^p, \mathbb{R})$ if $q \geq p$.

2.7 Involutions in the tensor product and representation cases

In Section 2.2 we discussed briefly the main and reversion involutions for the Grassmann algebra case, and how the later is related to the opposite algebra, while Theorem 3 provides the involutions for the ungraded tensor case. Describing a Clifford algebra $C\ell_{p,q}$ as a graded or ungraded tensor product, or using the Clifford valued matrix representation $\mathrm{Mat}(2, C\ell_{p,q})$, we want to describe in this Section how to construct the involutions and how they are implemented on these structures, see also [23].

The main (grade) involution of the Grassmann algebra $\bigwedge V$ induces the $\mathbb{Z}_2$-grading $\bigwedge V = \bigwedge_0 V + \bigwedge_1 V$. The ideal $\mathscr{I}_g$ generating the Clifford algebra $C\ell(V,g) = \bigwedge V / \mathscr{I}_g$ as a factor algebra is $\mathbb{Z}_2$-graded (even). Therefore, the $\mathbb{Z}_2$-grading is preserved without any change in the Clifford algebra. Hence, we can describe the main involution on the Grassmann basis

$$V^{\wedge,\flat} = \cup_{k \in \{0,\ldots,n\}} \{e_{i_1} \wedge \cdots \wedge e_{i_k} \mid I \in \{1,\ldots,n\}, i_l \in I, |I| = k\} \quad (24)$$

or the Clifford basis

$$V^{\&c,\flat} = \cup_{k \in \{0,\ldots,n\}} \{e_{i_1} \&c \cdots \&c\, e_{i_k} \mid I \in \{1,\ldots,n\}, i_l \in I, |I| = k\} \quad (25)$$

[8] The transversal $\{m_i\}$ is a set of Grassmann monomials in the basis of $C\ell_{p,q}$ which provide coset representatives of the quotient group $G_{p,q}/G_{p,q}(f)$ where $G_{p,q}(f)$ is the stabilizer group of a primitive idempotent f. $G_{p,q}(f)$ is a normal subgroup of the Salingaros vee group $G_{p,q} \subset C\ell_{p,q}$. For more, see [8].

underlying a Clifford algebra using the Grassmann $\mathbb{Z}_2$-grading.[9] That is, mapping the generators $e_i \mapsto -e_i$ in both cases. From the property of the Grassmann functor (7), by replacing (formally) $\hat{\ } : V \mapsto -V$ under the grade involution $\hat{\ }$, we derive

$$\bigwedge(-V-W) = \bigwedge(-V) \,\hat{\otimes}\, \bigwedge(-W).$$

This amounts to saying that $\hat{\ }|_{V+W} = \hat{\ }|_V \otimes \hat{\ }|_W$, and the same is true for the Clifford functor $C\ell$. The grade involution on graded and ungraded tensor products of Clifford algebras reads then:

$$\begin{aligned}\hat{\ } : C\ell_{p,q} \,\hat{\otimes}\, C\ell_{r,s} \to C\ell_{p,q} \,\hat{\otimes}\, C\ell_{r,s} :: v \,\hat{\otimes}\, w \mapsto \hat{v} \,\hat{\otimes}\, \hat{w},\\ \hat{\ } : C\ell_{p,q} \otimes C\ell_{r,s} \to C\ell_{p,q} \otimes C\ell_{r,s} :: v \,\hat{\otimes}\, w \mapsto \hat{v} \,\hat{\otimes}\, \hat{w}.\end{aligned} \tag{26}$$

The ungraded tensor behaves as the graded one since no generators need to be switched. The matrix case needs to implement the grade involution on the spinor representation and depends hence on the choice of the transversal $\{m_j\}$ and the idempotent f_i, see (23) and text afterwards. As the space V_2 in the decomposition in Theorem 3 is even, the grade involution can be obtained in an basis-invariant way by $[X] \mapsto [\omega][\hat{X}][\omega]^{-1}$ removing the dependency on a chosen basis. ($\hat{X}$ is the grade involution of $X \in C\ell_{p,q}$ and conjugating with $[\omega]$ is the grade involution in the spinor representation of $C\ell_{1,1}$.) For the basis (23) of Example 2 we get for the grade involution

$$\begin{aligned}\hat{\ } : \mathrm{Mat}(2, C\ell_{p,q}) \to \mathrm{Mat}(2, C\ell_{p,q}) :: \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} &\mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \hat{x}_{11} & \hat{x}_{12} \\ \hat{x}_{21} & \hat{x}_{22} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \\ \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} &\mapsto \begin{bmatrix} \hat{x}_{11} & -\hat{x}_{12} \\ -\hat{x}_{21} & \hat{x}_{22} \end{bmatrix}.\end{aligned} \tag{27}$$

We conclude that the three ways to describe product Clifford algebras, graded tensor product, ungraded tensor products from Theorem 2, and their Clifford valued matrix versions provide (grade) involutive Clifford algebra isomorphisms. The code for these maps is shown in Appendix 5.1.

The reversion involution interacts in a more involved way with the product algebras. We have seen in Section 2.2 that the reversion involution $\mathrm{rev}^{\wedge}$ relates the Grassmann algebra $\bigwedge V$ and its opposite algebra $\bigwedge^{op} V$. The same is true for Clifford algebras (of arbitrary bilinear forms B), $\mathrm{rev}^{\&c} : C\ell(V,B) \to C\ell^{op}(V,B)$. However, the reversion 'reverses' Clifford generators of the Clifford basis (25) and *not* that of the Grassmann basis (24). The interaction between reversion and Clifford product was presented in [6], in a categorical basis-free way. As one works for geometric reasons mostly with the Grassmann basis, we need to establish the reversion on this basis. Moreover, in case of a bilinear form B having a nontrivial nonsymmetric part, the reversion will depend on the antisymmetric part of the bilinear form B. As the reversion preserves the grade, it commutes with the grade involution.

---

9 In (25), $\&c$ denotes the Clifford product in $C\ell(B)$ as an ampersand operator in `CLIFFORD`.

We start with the graded tensor product case. Let `B=B1+B2` be a direct decomposition of the bilinear form on the linear space $V = V_1 + V_2$, hence $B|_V = B|_{V_1} + B|_{V_2} = B_1 + B_2$ (off diagonal entries need to be zero) and let rev_B, rev_{B1}, rev_{B2} be the respective reversions on $C\ell_{p+r,q+s}$, $C\ell_{p,q}$ and $C\ell_{r,s}$. The reversion map reads then

$$\mathrm{rev}_B : C\ell_{p,q} \hat{\otimes} C\ell_{r,s} \to C\ell^{op}_{p,q} \hat{\otimes} C\ell^{op}_{r,s} :: x \hat{\otimes} y \mapsto (\hat{\sigma} \circ \sigma)(\mathrm{rev}_{B_1}(x) \hat{\otimes} \mathrm{rev}_{B_2}(y)) \quad (28)$$

with $\hat{\sigma}$ being the graded switch and σ being the ungraded switch [8,9].

The ungraded tensor product cases from Theorem 2 have to cope with the fact that a switch of the tensor factors does not imply a sign factor. However, the map $(x,y) \mapsto x \otimes \omega + 1 \otimes y$ from Theorem 3 has to be taken into account. The reversion map has to be defined, for the decomposition $B = B_1 + B_2$, $B_{p+r,q+s} = B_{p,q} + B_{r,s}$ (for suitable r,s), as

$$\begin{aligned} \mathrm{rev}_B : (C\ell^0_{p,q} \oplus C\ell^1_{p,q}) \otimes C\ell_{r,s} &\to (C\ell^{0,op}_{p,q} \otimes C\ell^{op}_{r,s}) \oplus (C\ell^{1,op}_{p,q} \otimes \hat{C\ell}^{op}_{r,s}) \\ :: x \hat{\otimes} y &\mapsto (\mathrm{rev}_{B_1}(x^0) \otimes \mathrm{rev}_{B_2}(y)) + (\mathrm{rev}_{B_1}(x^1) \otimes \mathrm{rev}_{B_2}(\hat{y})). \end{aligned} \quad (29)$$

Here $x = x^0 + x^1 \in C\ell^0_{p,q} \oplus C\ell^1_{p,q}$ is the decomposition into even and odd parts, and $\hat{C\ell}_{p,q}$ is the grade involution of $C\ell_{p,q}$.

In the matrix case $\mathrm{Mat}(2, C\ell_{p,q})$ we need to investigate how the spinor representation implements the reversion on $C\ell_{1,1}$. Picking a basis, e.g., as in (23), picks a non natural isomorphism defining the spinor basis. However, we want to describe the reversion, if possible, without dependency on such choices. The matrix elements of an element x in a spinor basis for the idempotent f_s and transversal $\{m_i\}$ with basis $\{f^i_s = m_i f_s\}$ are given as $x_{ij} = \mathrm{tr}(f^{*,i}_s x f^j_s)$. Here, f^1_s, f^2_s are the spinor basis elements of (22), and $f^{*,1}_s = f^1_s, f^{*,2}_s = e_1 f^2_s$ is the dual basis with respect to the trace form. One can show that there is no element r in $C\ell_{1,1}$ such that $\mathrm{rev}_{1,1}(x) = rxr^{-1}$ due to the different action on even and odd elements. Using the transposition automorphism `tp`, which implements the matrix transposition as the transposition anti-involution $T_\varepsilon^{\sim}$ in $C\ell_{p,q}$ and which was discussed extensively in [8–10], we can use an invertible element $a = \mathtt{tp}(a)$ (represented by a matrix $[A]$ and we have $[\mathtt{tp}(a)] = [A]^t$) to model the reversion as follows

$$[\mathrm{rev}_{1,1}(X)] = [a\, \mathtt{tp}(X)\, a^{-1}] = [A][X]^t[A]^{-1}, \quad (30)$$

where $[X]^t$ is the matrix transpose of $[X]$ in $\mathrm{Mat}(2, C\ell_{p,q})$. For the basis of (23), we have $a = e_1$.

Concretely, for the basis in (22) based on an even idempotent $\hat{f}_1 = f_1$, we get the reversion map

$$\begin{aligned} \mathrm{rev}_{\mathrm{Mat}} &: \mathrm{Mat}(2, C\ell_{p,q}) \to \mathrm{Mat}(2, C\ell_{p,q}) \\ &:: \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} \mapsto \begin{bmatrix} \mathrm{rev}_{p,q}(\hat{x}_{22}) & \mathrm{rev}_{p,q}(\hat{x}_{12}) \\ \mathrm{rev}_{p,q}(\hat{x}_{21}) & \mathrm{rev}_{p,q}(\hat{x}_{22}) \end{bmatrix} \end{aligned} \quad (31)$$

where $\mathrm{rev}_{\mathrm{Mat}} = \mathrm{rev}_B$ is the reversion on $\mathrm{Mat}(2, C\ell_{p,q}) \cong C\ell_{p+1,q+1} \cong C\ell_{p,q} \otimes C\ell_{1,1}$ and $\mathrm{rev}_{p,q}$ is the reversion on $C\ell_{p,q}$. For a general spinor basis, one gets

$$\mathrm{rev}_B([X]) = \mathrm{rev}_B([X]^0 + [X]^1) = [a][\mathrm{rev}_{B_1}(X)]^0[a^{-1}] + [\mathrm{rev}_{B_1}(X)]^1 \tag{32}$$

with an invertible a satisfying $\mathtt{tp}(a) = a$. The code of all these grade and reversion involutions is displayed in Appendix 5.1 and is further discussed in the worksheets posted at [7].

3 Computing with `CLIFFORD` and `Bigebra` in tensor algebras

We provide examples of Maple code how to set up tensor products of Clifford algebras in `CLIFFORD` and `Bigebra`. For the usage of these packages see [1,5,6], the help pages which come with the package, and the package website [4]. The Maple worksheets with code for the described methods are posted at [7].

Loading the package using `>with(Clifford);with(Bigebra);` exposes the exported functions. To set up a Clifford algebra, say $C\ell_{2,2}$, one needs to define the dimension `>dim_V:=2+2;` and the bilinear form `>B:=linalg[diag](1$2,-1$2);`.[10] Basis elements e_I are written as strings, e.g., `e1we4` stands for $e_1 \wedge e_4$, etc., whereas `Id` stands for the identity of the Clifford algebra. `Bigebra` exports also the (graded) tensor product `&t`, which is multilinear and associative. Then, a tensor product $e_{1,2} \otimes e_1$ reads `&t(e1we2,e1)`, and permutations of tensors are implemented by maps `>switch(&t(e1,e2),1) = &t(e2,e1)` (the ungraded switch) or, in the graded case, `>gswitch(&t(e1,e2),1) = -&t(e2,e1)` (the graded switch). The extra index i in either switch (here we have used `1` in each) tells `[g]switch` to swap the i-th and the (i+1)st elements. Again, you get help by typing `>?switch` and `>?gswitch` at the Maple prompt.

The Clifford product `cmul` by default implicitly uses the bilinear form B as in, for example, `>cmul(e1,e2)=e1we2+B[1,2]*Id`. However, it can also use B or any other Maple name explicitly as an optional argument, e.g., `>cmul[K](e1,e2) = e1we2+K[1,2]*Id`, allowing to compute in different Clifford algebras in the same worksheet.

Let B, B_1, B_2 hold the bilinear forms[11] for $C\ell_{p+r,q+s}, C\ell_{p,q}$ and $C\ell_{r,s}$, and let `bas2GTbas` be the graded algebra isomorphism (16) given by $e_I \in C\ell_{p,q} \mapsto$ `&t(eI,Id)` ($I \subseteq \{1,\ldots,p+q\}$) and $e_J \in C\ell_{r,s} \mapsto$ `&t(Id,eJ)` ($J \subseteq \{1,\ldots r+s\}$), then the procedure `cmulGTensor` implements the Clifford algebra product in the *graded* tensor product of Clifford algebras in the r.h.s. of (16) as explained in Section 2.4.

---

[10] The default name of the bilinear form in `CLIFFORD` and `Bigebra` is B, however other names can also be used. So, when the bilinear form B is left undefined (unassigned), computations are performed in a Clifford algebra $C\ell(B)$ for an arbitrary bilinear form B (see, e.g., [6, 18]).

[11] Computations in the worksheet `cmulGTensor.mw` are performed for *arbitrary* not necessarily symmetric or diagonal bilinear forms.

```
cmulGTensor:=proc(x,y,B1,B2) local f4;
   f4:=(a,b,x,y)->cmul[B1](a,b),cmul[B2](x,y):
   eval(subs('&t'=f4,gswitch(&t(x,y),2)));
end proc:
```

We prove in the worksheet `cmulGTensor.mw` [7] that

```
GTbas2bas(cmulGTensor(X,Y,B1,B2))=
                              cmul[B](GTbas2bas(X),GTbas2bas(Y))
```

is the graded algebra isomorphism (16) with the inverse `bas2GTbas`. The grade and reversion involutions work as expected. The limit $B \to 0$ implements the wedge product on $\bigwedge V = \bigwedge V_1 \hat{\otimes} \bigwedge V_2$.

We discuss in more detail the *ungraded* tensor product for

$$C\ell_{p+1,q+1} \simeq C\ell_{p,q} \otimes C\ell_{1,1},$$

the isomorphism 6) of Theorem 2, which we call `bas2Tbas` while its inverse is `Tbas2bas`. We have in $C\ell_{1,1}$, with generators $e_1^2 = 1 = -e_2^2$, the volume element $\omega = e_1 e_2$ with $\omega^2 = \lambda = 1$, as we have to use the bilinear form $\frac{1}{\lambda} Q_1$ of Theorem 3, so we can still use $\frac{1}{\lambda} B_1 = B_1$. The isomorphism 6) reads $e_I \in C\ell_{p,q} \mapsto$`&t(eI,w)` ($I \subseteq \{1,\ldots,p+q\}$) and $e_J \in C\ell_{1,1} \mapsto$`&t(Id,eJ)` ($J \subseteq \{1,2\}$).[12] The tensor Clifford product is graded isomorphic to the Clifford product on $C\ell_{p+1,q+1}$. For a proof see the worksheet `cmulTensor11.mw` [7]. The following procedure implements the *ungraded* tensor product algebra as explained in Section 2.5.

```
cmulTensor:=proc(x,y,lB1,B2) local f4;
   f4:=(a,b,x,y)->cmul[lB1](a,b),cmul[B2](x,y):
   eval(subs('&t'=f4,switch(&t(x,y),2)));  # ordinary switch
end proc:
```

The isomorphism `bas2Tbas` with its inverse `Tbas2bas` this time is more involved, as are the grade and reversion involutions. We still get the isomorphism

```
Tbas2bas(cmulTensor(X,Y,B1,B2))=
                              cmul[B](Tbas2bas(X),Tbas2bas(Y))
```

proved by direct computation explicitly. Further details are provided in the worksheet. Note that we do not have the limit $B \to 0$, as $Q_2 \simeq B_2$ needs to be nondegenerate, and the naive limit replacing `cmul` by `wedge` gives false results. In the worksheet we show how to produce the Grassmann basis for the tensor algebra, and how the isomorphism and the involutions work.

In each worksheet we have benchmarked computations using the generic `cmul` routine for the Clifford product from `CLIFFORD` (in $\dim V \leq 9$) versus `Bigebra` tensor routines `cmul[G]Tensor`. For orthonormal bases we roughly get equal runtimes. The more complicated data structures of the tensor algebras is compensated by not computing some of the off-diagonal terms.

We add to this brief discussion that it is easily possible to iterate this morphism, and to provide `CLIFFORD` code for computations in Clifford algebras

[12] Here, `w` in `&t(eI,w)` stands for the volume element ω. Then, in the code of `cmulTensor`, the form $\frac{1}{\lambda} B_1$ is denoted as `lB1`.

$$C\ell_{p+k,q+k} \simeq C\ell_{p,q} \otimes \underbrace{C\ell_{1,1} \otimes \cdots \otimes C\ell_{1,1}}_{k \text{ factors}}, \tag{33}$$

or use the $\bmod 8$ periodicity.

4 Computations using matrix algebras over Clifford numbers

The isomorphism 6) from Theorem 2 was explicitly defined by Lounesto in [21, Sect. 16.3]. We will use this matrix approach to perform computations in $C\ell_{8,2} \simeq \mathrm{Mat}(2, C\ell_{7,1})$ [11]. Let $\{e_1, \ldots, e_8\}$ be an orthonormal basis of $\mathbb{R}^{7,1}$ generating the Clifford algebra $C\ell_{7,1}$ such that $e_i^2 = 1$ for $1 \leq i \leq 7$, $e_8^2 = -1$, and $e_i e_j = -e_j e_i$ for $\leq i, j \leq 8$ and $i \neq j$. The following 2×2 matrices (compare with (23))

$$E_i = \begin{pmatrix} e_i & 0 \\ 0 & -e_i \end{pmatrix} \quad \text{for} \quad i = 1, \ldots, 8, \quad E_9 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad E_{10} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{34}$$

anti-commute and generate $C\ell_{8,2}$.[13] In order to effectively compute in $C\ell_{8,2}$, we define the isomorphism $\varphi : \mathrm{Mat}(2, C\ell_{7,1}) \to C\ell_{8,2}$ as a Maple procedure `phi` from the `asvd` package.[14] Its inverse is given by the Maple procedure `evalm`. In the first step, we compute 1,024 2×2 matrices E_I with entries in $C\ell_{7,1}$ which represent the basis monomials $e_I = e_{i_1} e_{i_2} \cdots e_{i_k}$, $i_1 < i_2 < \cdots < i_k$, $0 \leq k \leq 8$. We store these matrices in a list $\mathscr{B}$.

For example, for the basis $\mathscr{B}_k$ of k-vectors in `CLIFFORD` we compute all $\binom{8}{k}$ products E_{i_1} `&cm` E_{i_2} `&cm` $\cdots$ `&cm` E_{i_k} where `&cm` is a `CLIFFORD` procedure to compute a product of Clifford algebra-valued matrices with the Clifford product applied to the matrix entries. Once the matrix representation (34) has been chosen,[15] the list $\mathscr{B} = \bigcup_k \mathscr{B}_k$ can be saved and read into a next Maple session thus avoiding the need to repeat this step.

Having computed the basis matrices $\mathscr{B}$, we are now ready to compute in $C\ell_{8,2}$. For example,[16] let $x = 2ID + 4E_{1,2,3} - 10E_{1,5,7,8,10}$ and $y = -ID + 4E_{1,2,3,7} + E_{1,5,6,8} - 3E_{1,4,6,7}$. We can find the Clifford product x `&CM` y of x and y in $C\ell_{8,2}$ using the following procedure:

```
`&CM`:=proc(x::algebraic,y::algebraic) local xy; global phi,BBB;
  if not type(evalm(x),climatrix) then error
     `evalm(x) must be of type climatrix` end if:
```

13 Notice that $E_i^2 = E_9^2 = 1$ for $1 \leq i \leq 7$ and $E_8^2 = E_{10}^2 = -1$ in this explicit representation. Also, $C\ell_{1,1} \simeq \mathrm{Mat}(2, \mathbb{R})$ is such that the matrices $e_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $e_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ generate $C\ell_{1,1}$.

14 The `asvd` package is part of the `CLIFFORD` library. It was introduced in [3].

15 Note that other representations are possible. See [21, Sect.16.3].

16 In the following, we let ID denote the identity matrix in $\mathrm{Mat}(2, C\ell_{7,1})$ namely $ID = \begin{pmatrix} \mathtt{Id} & 0 \\ 0 & \mathtt{Id} \end{pmatrix}$ where `Id`, as before, denotes the identity element in $C\ell_{7,1}$.

```
  if not type(evalm(y),climatrix) then error
     `evalm(y) must be of type climatrix` end if:
  xy:=displayid(evalm(x) &cm evalm(y));
  return phi(xy,BBB);
end proc:
```

Once the basis $\mathscr{B}$ is stored in Maple as a list `BBB`, the procedure `&CM` treats it as a global variable and it returns[17]

$$-40E_{2,3,5,8,10}+2E_{1,5,6,8}-16E_{7}-4E_{1,2,3}+10E_{1,5,7,8,10}-10E_{6,7,10} \\ +8E_{1,2,3,7}+4E_{2,3,5,6,8}-12E_{2,3,4,6,7}-30E_{4,5,6,8,10}-6E_{1,4,6,7}-2ID\,. \quad (35)$$

Details of the above computations can be found in the worksheet `G82.mws` [7]. In the worksheet we also define the respective grade and reversion involutions, and the graded algebra isomorphisms $C\ell_{8,2} \leftrightarrow \mathrm{Mat}(2, C\ell_{7,1})$ are given by `phi` and `evalm`.

5 Appendix

5.1 *Code for the involutions for product and matrix Clifford algebras*

The grade involutions are coded as follows:

Listing 1 Graded main involution

```
# GTgradeinv : grade involution on CL_p,q (x) CL_r,s
GTgradeinv:=proc(x) local f2;
  f2:=(a,b)->&t(gradeinv(a),gradeinv(b));
  eval(subs(`&t`=f2,x));
end proc:
```

The local function `f2` maps the grade involution to the tensor arguments, and the `eval(subs...)` line applies the tensor (triggering multilinearity). This simple code reflects the simple definition of the main involution as discussed in Section 2.7. The ungraded case works out exactly the same way:

Listing 2 Ungraded main involution

```
# Tgradeinv : grade involution on CL_p,q (x) CL_r,s
Tgradeinv:=proc(x) local f2;
  f2:=(a,b)->&t(gradeinv(a),gradeinv(b));
  eval(subs(`&t`=f2,x));
end proc:
```

[17] On a laptop running Intel(R) Core(TM) 2 Duo CPU T6670 @ 2.20 GHz it takes 6.5 sec to obtain this result. The computation time can be shortened using parallel processing available in Maple 15 and later.

The $\mathrm{Mat}(2,C\ell_{p,q})$ case is different. Due to the choice of a spinor basis for $C\ell_{1,1}$, the grade involution depends on this choice. Using the basis defined in Section 2.6 equation (23), we code the graded involution as

Listing 3 $\mathrm{Mat}(2,C\ell_{p,q})$ main involution

```
# Mgradeinv : grade involution on Mat(2,CL_p,q)
Mgradeinv:=proc(x)
  linalg[matrix](2,2, [ gradeinv(x[1,1]),-gradeinv(x[1,2]),
                       -gradeinv(x[2,1]), gradeinv(x[2,2])]);
end proc:
```

This reflects the fact that in this spinor basis the non zero diagonal terms (e_i) of generators are odd, while the non zero off diagonal terms are even (± 1) and need an additional minus sign.

The reversion is more complicated as it involves swapping of generators between the two factors of the product representations or involves the chosen spinor basis. The graded tensor case just needs an additional sign due to the swapping of the two factors of the product:

Listing 4 Graded reversion

```
# GTreversion : reversion involution on CL_p,q (x) CL_r,s
#               !! works for general bilinear forms B1 & B2 !!
GTreversion:=proc(x,B1,B2) local f2;
   f2:=(a,b)->&t(reversion(b,B1),reversion(a,B2)); # note order
                                                   # of a,b
   eval(subs(`&t`=f2,gswitch(x,1))); # gswitch for correct sign
end proc:
```

The reversion in the ungraded case is done by identifying two cases. If the first case, the tensor factor is even, the reversion is an (ungraded) tensor product morphism. In the odd case, one needs to apply additionally the grade involution to the first factor. This is done in the local function `f2`, which is then applied to all tensor monomials (using the `eval(subs(`&t`=...))` evaluation of the tensor).

Listing 5 Ungraded reversion

```
# Treversion : reversion involution on CL_p,q (x) CL_r,s
#              !! works for general bilinear forms B1 & B2 !!
Treversion:=proc(x,B1,B2) local f2;
   f2:=proc(a,b)
     if nops(Clifford:-extract(a)) mod 2 = 0 then
       return &t(reversion(a,B1),reversion(b,B2));
     else
       return &t(gradeinv(reversion(a,B1)),(reversion(b,B2)));
     end if;
   end proc:
   eval(subs(`&t`=f2,x));
end proc:
```

The reversion in the $\mathrm{Mat}(2, C\ell_{p,q})$ case depends on the basis chosen in (23). It swaps the diagonal entries and has to apply the grade involution to the second column.[18]

Listing 6 $\mathrm{Mat}(2, C\ell_{p,q})$ reversion

```
#  Mreversion : reversion on Mat(2,CL_p,q)
#        NOTE: depends on spinor basis for CL_1,1
Mreversion:=proc(x,B)
 linalg[matrix](2,2,
  [gradeinv(reversion(x[2,2],B)),gradeinv(reversion(x[1,2],B)),
   gradeinv(reversion(x[2,1],B)),gradeinv(reversion(x[1,1],B))]);
end proc:
```

References

1. Abłamowicz, R.: Clifford algebra computations with Maple. In: Baylis, W. E. (ed.) Clifford (Geometric) Algebras with Applications in Physics, Mathematics, and Engineering, pp. 463–502, Birkhäuser, Boston (1996)
2. Abłamowicz, R.: Spinor representations of Clifford algebras: A symbolic approach. Computer Physics Communications Thematic Issue "Computer Algebra in Physics Research" Vol. **115**, Numbers 2–3, 510–535 (1998)
3. ———: Computations with Clifford and Grassmann algebras. Adv. Applied Clifford Algebras **19**, No. 3–4, 499–545 (2009)
4. Abłamowicz, R., Fauser, B.: `CLIFFORD` with `Bigebra` – A Maple Package for Computations with Clifford and Grassmann Algebras (2012), `http://math.tntech.edu/rafal/`. Cited June 10, 2012
5. ———: Clifford and Grassmann Hopf algebras via the `Bigebra` package for Maple. Computer Physics Communications **170**, 115–130 (2005), math-ph/0212032
6. ———: Mathematics of `CLIFFORD` – A Maple Package for Clifford and Grassmann Algebras. Adv. in Applied Clifford Algebras **15**, No. 2, 157–181 (2005), math-ph/0212031
7. ———: Maple worksheets created with `CLIFFORD` for verification of the results presented in this paper (2012), `http://math.tntech.edu/rafal/MapleWorksheets3.html`. Cited June 10, 2012
8. ———: On the transposition anti-involution in real Clifford algebras III: the automorphism group of the transposition scalar product on spinor spaces. Linear and Multilinear Algebra Vol. **60**, No. 6, 621-644 (2012)
9. ———: On the transposition anti-involution in real Clifford algebras II: stabilizer groups of primitive idempotents. Linear and Multilinear Algebra, Vol. **59**, No. 12, 1359–1381 (2011)
10. ———: On the transposition anti-involution in real Clifford algebras I: the transposition map. Linear and Multilinear Algebra, Vol. **59**, No. 12, 1331–1358 (2011)
11. Bayro-Corrochano, E.: Private communication (2012)
12. Anglés, P: The structure of the Clifford algebra. Adv. Applied Clifford Algebras **19**, No. 3–4, 585–610 (2009)
13. Atiyah, M. F., Bott, R., Shapiro, A.: Clifford modules. Topology Vol. **3**, Suppl. 1, 3–38 (1964)
14. Budinich, P., Trautman, A.: The Spinorial Chessboard. Springer-Verlag, Berlin (1988)
15. Cartan, E. (exposé d'après l'article allemand de E. Study): Nombres complexes. In Molk, J. (ed.), Encyclopédie des sciences mathématiques, pp. 329–468, Tome **I**, Vol. **1**, Fasc. **4**, art. **15** (1908). Reprinted in Cartan, E.: Œuvres complètes, pp. 107–246, Partie **II**, Gauthier-Villars, Paris (1953)

[18] Columns correspond to spinor bases of isomorphic modules, however, the second column has a different parity.

16. Coquereaux, R.: Clifford Algebras, spinors and fundamental interactions: Twenty Years Later. Adv. in Applied Clifford Algebras **19**, No. 3–4, 673–686 (2009)
17. Crumeyrolle, A.: Orthogonal and Symplectic Clifford Algebras, Spinor Structures. Kluwer (2010)
18. Fauser, B., Abłamowicz, R.: On the decomposition of Clifford algebras of arbitrary bilinear form. In: Abłamowicz, R., Fauser, B. (eds.) Clifford Algebras and their Applications in Mathematical Physics, pp. 341–366, Birkhäuser, Boston (2000), math.QA/9911180
19. Helmstetter, J., Micali, A.: Quadratic Mappings and Clifford Algebras. Birkhäuser, Boston (2008)
20. Lam, T. Y.: The Algebraic Theory of Quadratic Forms, The Benjamin/Cummings, Reading, MA (1973)
21. Lounesto, P.: Clifford Algebras and Spinors, 2nd ed. Cambridge University Press, Cambridge (2001)
22. Mac Lane, S.: Categories for the Working Mathematician, 2nd ed. Springer-Verlag, New York (1978)
23. Maks, J.: Modulo (1,1) periodicity of Clifford algebras and the generalized (anti-) Möbius transformations. Ph.D. Thesis, TU Delft (1989)
24. Porteous, I.: Clifford Algebras and the Classical Groups. Cambridge University Press, Cambridge (1995)