DEPARTMENT OF MATHEMATICS

TECHNICAL REPORT

The Genre of Proof

ANNIE SELDEN

JOHN SELDEN

MARCH 2013

No. 2013-1

TENNESSEE TECHNOLOGICAL UNIVERSITY

Cookeville, TN 38505

THE GENRE OF PROOF[1]

Annie Selden and John Selden[2]
New Mexico State University

Students find many aspects of mathematical proof confusing. One area that they find especially perplexing is the manner in which proofs are written, which is often at variance with other genres of writing. Nardi and Iannone (2006) claimed that mastering proof involves acquiring a different genre of communication.

In the mathematics education literature, a variety of genres of mathematical writing have been considered. For example, in discussing mathematical writing at the school level, Marks and Mousley (1990, p. 119) distinguished narrative genre, procedural genre, description and report genres, exploratory genre, and expository genre. While all these have a place when examining mathematical writing more generally, in this short contribution, we restrict our considerations to the genre of proof, and in particular, to how mathematicians write proofs for publication. As Konior (1993) argued, studying the genre of mathematical proof is particularly important for mathematics educators, as this can inform how students should read and write proofs.

Mathematics educators, mathematicians, and philosophers have written about the genre of mathematical proof, emphasizing it special nature, its long evolution, and the impossibility of making it entirely explicit. For example, Ernest (1998, p. 169) stated, “Mathematical proof is a special form of text, which since the time of the ancient Greeks, has been presented in monological [rather than dialogical] form.” Jaffe (1990, p. 146) asserted that “The standards of what constitutes a proof have evolved over hundreds of years; there is no doubt in the minds of traditional mathematicians what a proof means.” Furthermore, according to Kitcher (1984, p. 163), in mathematical practice both tacit knowledge, or “know how,” and meta-mathematical views (including standards for proof) are important, and it is not possible for those standards to be made fully explicit. However, it may be possible to identify some significant features that generally occur in the genre of proofs.

A Study of Mathematicians’ Views on Features of Proofs

While at the Park City Mathematics Institute[3] some years ago, we asked mathematicians what they thought of the following seven conjectured features of proofs, while looking at one of their own published mathematics papers, and they tended to agree. Below we detail these seven features and provide supporting evidence from mathematicians, as well as some examples of observed university students’ violations of these features.

---

[1] This paper is based on a panel presentation on proof given at the Symposium on Mathematics Education, on the occasion of Ted Eisenberg’s 70th birthday, held at Ben Gurion University in Beer Sheva, Israel, April 29-May 3, 2012.

[2] Annie Selden is Professor Emerita of Mathematics from Tennessee Technological University. John Selden is retired from and also formerly taught mathematics at Tennessee Technological University. The Seldens have retired to Las Cruces, New Mexico and are now Adjunct Professors of Mathematics at New Mexico State University.

[3] The Park City Mathematics Institute is a program of the Institute for Advanced Study, Princeton, NJ. It is designed for mathematics researchers, post-secondary students, and mathematics educators at the secondary and post-secondary levels.

1. Proofs are not reports of the proving process.

For example, one does not write into a final written proof things like, "I tried this [technique or idea] and it did not work." One also does not (usually) write. "I want to show [the conclusion or sub-conclusion that one will prove next]," except perhaps in rather long, complicated proofs. Mathematicians do not consider "the function of the written document as a record of the work done … and consistency with historical development is hardly considered at all as a relevant factor ...". (Csiszar, 2003, pp. 247-8). "False starts, mistakes, revisions—these are all part of the creative process [in discovering and proving theorems]. But when the final result is published, we seldom see the enormous effort that was necessary for the creation; we see the polished product, the correct statement with a clean proof. … [This is] an important feature of mathematics." (Ewing, 1984, p. 3).

2. Proofs contain little redundancy.

Unlike arguments in philosophical papers, one does not consider the argument from another point of view (at least in the same proof). Furthermore, " … mathematicians appear to prize brevity, conciseness, and precision of meaning." (Shepherd, Selden, & Selden, 2012). In Halmos' (1970) *How to Write Mathematics*, there is a section titled, "Down with the Irrelevant and Trivia." There he said of the statements of theorems, and presumably the same holds for their proofs, that:

> … [T]here, more than anywhere else, irrelevancies must be avoided. … Leave the chit-chat out; "Without loss of generality, we may assume …" and 'Moreover, it follows from Theorem 1 that …" do not belong in the statement of a theorem. (pp. 138-139).

3. Symbols are (generally) introduced into proofs in one-to-one correspondence with mathematical objects.

"In the genre of mathematical proofs it is not permissible to let the same symbol represent two different numbers, except across independent subproofs. Perhaps this is because doing so seems very likely to cause validators confusion." (Selden & Selden, 2003, p. 13).

For example, one does *not* do what one of our transition-to-proof course students did, when proving the following **Theorem:** *For integers m, n, and p, if m divides n and m does not divide p, then n does not divide p.* The attempted proof began with the following unnecessary profusion of letters: **Proof:** *Since m , n, and p are integers and m divides n and n does not divide p, let m = j and n = jk and p = l, where j, k, and l are integers,* and did not get better.

4. Proofs contain only minimal explanations of inferences, that is, warrants are often left implicit.

The *Manual for Authors of Mathematical Papers* (AMS, 1962) suggests that authors, "Omit any computation which is routine (i.e., does not depend on unexpected tricks). Merely indicate the starting point, describe the procedure, and state the outcome…". Also, the Associate Editor for the *Journal of Geometric Analysis*, suggests that while most mathematical arguments need to be justified, sometimes "the reason will be so obvious to the reader that it is actually more effective to leave it out." (Lee, 2012, p. 3).

For example, one of our transition-to-proof students correctly proved the **Theorem:** For all sets *A* and *B*, if $A \cap B = A$, then $A \cup B = B$ as follows. **Proof:** Let *A* and *B* be sets. Suppose $A \cap B = A$. Let $x \in A$. Since $A = A \cap B$, then $x \in B$. [A warrant for this step would be: From $x \in A$ and $A = A \cap B$, one has $x \in A \cap B$. Then $x \in A$ and $x \in B$, so $x \in B$. But it was omitted.]

5. Proofs contain only very short overviews or advance organizers.

For example, one might say at the beginning of a proof by contradiction “Suppose to the contrary that …” However, one would not generally give the overall organization of the proof in advance, such as indicating that there would be a case argument, and that the first case would be proved directly.

This contravenes Leron’s (1983) idea of using a hierarchical structure of levels when *presenting* proofs in the classroom. He said, “While the age-old and venerable method [of presenting proofs in a step-by-step, “linear” fashion] may be well suited for securing the validity of proofs, it is nonetheless unsuitable for … presentations.”

6. Entire definitions (available outside the proof) are not quoted in proofs.

In a point-set topology proof, one would not usually include standard definitions of compact and connected. Similarly, Halmos (1970, p. 143) wrote, “If a reader knows what a sequence is, if he feels the definition in his bones, then …” including its definition would “distract him and slow his reading down, if ever so slightly”. However, it is quite common for beginning undergraduate real analysis students to state the entire definition of continuity within a proof, rather than simply writing, “By definition of continuity, we have …”.

Moore (1994, pp. 258-259) reported that when one transition-to-proof course student was asked on a test to prove: *If* A *and* B *are sets satisfying* $A \cap B = A$, *then* $A \cup B = B$, she drew a Venn diagram with one circle, labeled A, contained in a larger circle, labeled B, and gave an intuitive argument, whereas the professor “wanted a proof based only on definitions, axioms, previously proved results, and rules of inference.” Without understanding that there is a genre of proof, such obvious theorems may be very difficult for students to write in the expected way.

7. ***Proofs are "logically concrete" in the sense that quantifiers, especially universal quantifiers, are avoided where possible***.

Students often want to argue for all *x* (or for all ε) within a proof, rather than selecting a fixed, but arbitrary *x* (or ε) and arguing about that. It is implicit that, whichever element is selected, the argument can be about that element. Although this is a simple rhetorical device, it is very powerful and it simplifies the logic required of validators, thereby perhaps avoiding some errors.

Mary was a returning grad student taking beginning real analysis with Dr. K, who assigned 3 or 4 weekly proofs, graded them very thoroughly, and allowed them to be resubmitted. He emphasized things like writing “Let *x* be a number” into proofs. She recalled feeling this requirement was not particularly important or appropriate. She complied to get full credit. Near the middle of the course, Mary came to feel that this “made sense and it was the way to do it.” She reported to us, two years later, that she could not think of any other way to write (this aspect of) proofs. (Selden, McKee, & Selden, 2010).

Conclusion

The genre of proofs has developed over considerable time, but not necessarily by conscious intent. This is probably because mathematicians see this genre as having value. Csiszar (2003, p. 268) suggested that this rhetoric contributes to a sense of a “proof’s inevitability” and that readers can obtain pleasure from “seeing a proof unfold as it must before one’s [their] eyes.” In addition, sometimes mathematicians just want to get the flow of the argument without wallowing in the details. As the *MAMP* stated, readers want “to see the path—not examine it with a microscope.” Finally, we conjecture that one reason this genre may have developed is because it makes the validation of proofs as easy, and hence as reliable, as possible. That is, distractions are minimized, thereby maximizing the finding of errors.

References

AMS (1962). Manual for authors of mathematical papers. *Bulletin of the American Mathematical Society*, *68*(5), 429-444.

Csiszar, A. (2003). Stylizing rigor: Or, why mathematicians write so well. *Configurations*, *11*(2), 239-268.

Ernest, P. (1998). *Social constructivism as a philosophy of mathematics*. Albany: State University of New York Press.

Ewing, J. (1984). Editorial: A breach of etiquette. *The Mathematical Intelligencer*, *6*(4), 3-4.

Halmos, P. (1970). How to write mathematics. *L’Enseignment Mathématique*, *16*, 123-152.

Jafffe, A. (1990). Mathematis motivated by physics. In J. Glimm, J. Impagliazzo, & I. M. Singer (Eds.), *Proceedings of the Symposia in Pure Mathematics, Vol. 50: The legacy of John von Newuman* (pp. 137-152). Providence, RI: American Mathematical Society.

Kitcher, P. (1984). *The nature of mathematical knowledge*. Oxford: Oxford University Press.

Konior, J. (1993). Research into the construction of mathematical texts. *Educational Studies in Mathematics, 24*(3), 251-256.

Lee, J. L. (2012). Some remarks on writing mathematical proofs. Downloaded June 3, 2012 from http://www.math.washington.edu/~lee/Writing/writing-proofs.pdf.

Leron, U. (1983). Structuring mathematical proofs. *The American Mathematical Monthly*, *90*(3), 174-185

Marks, G., & Mousley, J. (1990). Mathematics education and genre: Dare we make the process writing mistake again? *Language and Education*, *4*(2), 117-135.

Moore, R. C. (1994). Making the transition to formal proof. *Educational Studies in Mathematics*, 27, 249-266.

Nardi E., & Iannone, P. (2006). To appear and to be: Mathematicians on their students’ attempts at acquiring the ‘genre speech’ of university mathematics. *Proceedings of the 4th Conference on European Research in Mathematics Education*, Sant Feliu de Guixols, Spain, *14*, 1800-1810.

Selden, A., McKee, K., & Selden, J. (2010). Affect, behavioural schemas and the proving process. *International Journal of Mathematical Education in Science and Technology*, *41*(2), 199-215.

Shepherd, M. D., Selden, A., & Selden, J. (2012). University students’ reading of their first-year mathematics textbooks. *Mathematical Thinking and Learning*, *14*, 226-256.