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# Bilinear Covariants and Spinor Field Classification in Quantum Clifford Algebras

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**Abstract.** In this letter, the Lounesto spinor field classification is extended to the spacetime quantum Clifford algebra and the associated quantum algebraic spinor fields are constructed. In order to accomplish this extension, the spin-Clifford bundle formalism is employed, where the algebraic and geometric objects of interest may be considered. In particular, we describe the bilinear covariant fields in the context of algebraic spinor fields. By endowing the underlying spacetime structure with a bilinear form that contains an antisymmetric part, which extends the spacetime metric, the quantum algebraic spinor fields are exhibited and compared to the standard case, together with the bilinear covariants that they induce. Quantum spinor field classes are hence introduced and a correspondence between them and the Lounesto spinor field classification is provided. A physical interpretation regarding the deformed parts and the underlying $\mathbb{Z}_n$-grading is also given. The existence of an arbitrary bilinear form endowing the spacetime, already explored in the literature in the context of quantum gravity [1], plays a prominent role in the structure of the Dirac, Weyl, and Majorana spinor fields, besides the most general flagpoles and flag-dipoles, which are shown to be capable to probe interesting features associated to the spacetime.

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## 1. Introduction

The formalism of Clifford algebras allows wide applications, in particular, the prominent construction of spinors and Dirac operators, and index theorems. Usually such algebras are essentially associated to an underlying quadratic vector space. Notwithstanding, there is nothing that complies to a symmetric bilinear form endowing the vector space [2]. For instance, symplectic Clifford algebras are objects of huge interest. More generally, when one endows the underlying vector space with an arbitrary bilinear form, it evinces prominent features, especially regarding their representation theory. The most drastic character distinguishing the so called quantum and the orthogonal Clifford algebras ones is that a different $\mathbb{Z}_n$-grading arises, despite of the $\mathbb{Z}_2$-grading being the same, since they are functorial. The most general Clifford algebras of multi-vectors [3] are further named quantum Clifford algebras. The arbitrary bilinear form that defines the quantum Clifford algebra defines a $\mathbb{Z}_n$-grading, which are not solely suitable, instead hugely necessary. For instance, it is employed in every quantum mechanical setup. In fact, when one analyzes functional hierarchy equations of quantum field theory, one is able to use Clifford algebras to emulate the description of these functionals. At least the time-ordering and normal-ordering are needed in quantum field theory [4], and singularities due to the reordering procedures such as the normal-ordering, are no longer present [5]. Quantum Clifford algebras can be led to Hecke algebras in a very particular case [6] and this structure should play a major role in the discussion of the Yang-Baxter equation, the knot theory, the link invariants and in other related fields which are crucial for the physics of integrable systems in statistical physics. Moreover, the coalgebraic and the Hopf algebraic structure associated to those algebras can be completely defined [7, 8].

From the physical point of view, there can be found a list of references containing a non symmetric gravity theory, and some applications. For instance, the rotation curves of galaxies and cosmology, without adducing dominant dark matter and identifying dark energy with the cosmological constant, can be obtained [9, 10], by just considering the spacetime metric to have symmetric *and* antisymmetric parts. Besides, a gravitational theory based on general relativity was formulated and discussed in [9, 10], concerning non symmetric tensors playing the role of the spacetime metric.

This paper aims to provide a complete classification of algebraic spinor fields in quantum Clifford algebras, also in the context of their representations. It is organized as follows: in Section 2, the Clifford and spinor bundles are revisited; in Section 3, the bilinear covariants and the associated spinor field classification are recalled; and in Section 4, the correspondence between the classical and the algebraic spinor fields is obtained. In Section 5, the quantum algebraic spinor fields and some important properties are introduced and investigated. In Section 6, the spinor field classification, according to their bilinear covariants, is accomplished in the quantum Clifford algebraic formalism. The spinor field disjoint classes that encompass the Dirac, Weyl, Majorana – the flagpoles – and the flag-dipoles are shown to reveal a dramatic alterations with potentially prominent physical applications. In Appendix A we show explicitly a complete set of primitive and orthogonal idempotents in $\mathbb{C}\otimes C\ell_{1,3}^{B}$ and a spinor basis in a minimal ideal $S_B = (\mathbb{C}\otimes C\ell_{1,3}^{B})f_B$ while in Appendix B we calculate each part of a $B$-spinor $\underset{B}{(\psi_B)}(f_B) \in S_B$.

## 2. Preliminaries

Let $V$ be a finite $n$-dimensional real vector space and $V^*$ denotes its dual. The exterior algebra $\bigwedge V = \oplus_{k=0}^{n}\bigwedge^k V$ is the space of the antisymmetric $k$-tensors. Given $\psi\in\bigwedge V$, the reversion is given by $\tilde{\psi} = (-1)^{[k/2]}\psi$, where $[k]$ corresponds to the integer part of $k$. If $V$ is endowed with a non-degenerate, symmetric, bilinear map $g: V\times V\to\mathbb{R}$, it is possible to extend $g$ to $\bigwedge V$. Given $\psi = \mathbf{a}^1\wedge\cdots\wedge\mathbf{a}^k$ and $\phi = \mathbf{b}^1\wedge\cdots\wedge\mathbf{b}^l$, for $\mathbf{a}^i, \mathbf{b}^j\in V$, $g(\psi,\phi) = \det(g(\mathbf{a}^i,\mathbf{b}^j))$ if $k=l$ and $g(\psi,\phi)=0$ if $k\neq l$. Given $\psi,\phi,\xi\in\bigwedge V$, the left contraction is defined implicitly by $g(\psi\lrcorner\phi,\xi) = g(\phi,\tilde{\psi}\wedge\xi)$. The right contraction is analogously defined by $g(\psi\llcorner\phi,\xi) = g(\phi,\psi\wedge\tilde{\xi})$. The Clifford product between $\mathbf{w}\in V$ and $\psi\in\bigwedge V$ is given by $\mathbf{w}\psi = \mathbf{w}\wedge\psi + \mathbf{w}\llcorner\psi$. The Grassmann algebra $(\bigwedge V, g)$ endowed with the Clifford product is denoted by $C\ell(V,g)$ or $C\ell_{p,q}$, the Clifford algebra associated with $V\simeq\mathbb{R}^{p,q}$, $p+q=n$.

By restricting to the case where $p=1$ and $q=3$, the Clifford and spin-Clifford bundles are briefly revisited [11, 12]. Denote by $(M, g, \nabla, \tau_g, \uparrow)$ the spacetime structure: $M$ is a 4-dimensional manifold, $g\in\sec T_2^0M$ is the metric of the cotangent bundle (in an arbitrary basis $g = g_{\alpha\beta}dx^\alpha\otimes dx^\beta$), $\nabla$ is the Levi-Civita connection of $g$, $\tau_g\in\sec\bigwedge^4 TM$ defines a spacetime orientation and $\uparrow$ is an equivalence class of timelike 1-form fields defining a time orientation. $F(M)$ denotes the principal bundle of frames, $\mathbf{P}_{\mathrm{SO}_{1,3}^e}(M)$ is the orthonormal frame bundle, and $P_{\mathrm{SO}_{1,3}^e}(M)$ the orthonormal coframe bundle. Moreover, when $M$ is a spin manifold, there exists $\mathbf{P}_{\mathrm{Spin}_{1,3}^e}(M)$ and $P_{\mathrm{Spin}_{1,3}^e}(M)$, respectively called the spin frame and the spin coframe bundles. By denoting $r: P_{\mathrm{Spin}_{1,3}^e}(M)\to P_{\mathrm{SO}_{1,3}^e}(M)$ the mapping in the definition of $P_{\mathrm{Spin}_{1,3}^e}(M)$, a spin structure on $M$ is constituted by a principal bundle $\pi_r: P_{\mathrm{Spin}_{1,3}^e}(M)\to M$, with group $\mathrm{Spin}_{1,3}^e$, and the map $r: P_{\mathrm{Spin}_{1,3}^e}(M)\to P_{\mathrm{SO}_{1,3}^e}(M)$ satisfying:

(i) $\pi(r(p)) = \pi_r(p)$, $\forall p\in P_{\mathrm{Spin}_{1,3}^e}(M)$, where $\pi$ is the projection map of the bundle $P_{\mathrm{SO}_{1,3}^e}(M)$.

(ii) $r(p\phi) = r(p)\mathrm{ad}_\phi$, $\forall p\in P_{\mathrm{Spin}_{1,3}^e}(M)$ and $\mathrm{ad}: \mathrm{Spin}_{1,3}^e\to\mathrm{Aut}(C\ell_{1,3})$, $\mathrm{ad}_\phi: \omega\mapsto\phi\omega\phi^{-1}\in C\ell_{1,3}$ [11, 12].

Sections of $P_{\mathrm{SO}_{1,3}^e}(M)$ are orthonormal coframes, and the ones in $P_{\mathrm{Spin}_{1,3}^e}(M)$ are also orthonormal coframes — two coframes differing by a $2\pi$ rotation are distinct and two coframes differing by a $4\pi$ rotation are equivalent. The Clifford bundle of differential forms $C\ell(M,g)$ is a vector bundle associated to $P_{\mathrm{Spin}_{1,3}^e}(M)$, having as sections sums of differential forms — Clifford fields. Furthermore, $C\ell(M,g) = P_{\mathrm{SO}_{1,3}^e}(M)\times_{\mathrm{ad}'} C\ell_{1,3}$, where $C\ell_{1,3}\simeq\mathrm{Mat}(2,\mathbb{H})$ is the spacetime algebra. The bundle structure is obtained as:

(a) Consider $\pi_c : C\ell(M,g) \to M$ be the canonical projection and $\{U_\alpha\}$ is an open covering of $M$. There are trivialization mappings $\psi_i : \pi_c^{-1}(U_\alpha) \to U_\alpha \times C\ell_{1,3}$ of the form $\psi_\alpha(p) = (\pi_c(p), \psi_{\alpha,x}(p)) = (x, \psi_{\alpha,x}(p))$. If $x \in U_\alpha \cap U_\beta$ and $p \in \pi_c^{-1}(x)$, therefore

$$\psi_{\alpha,x}(p) = h_{\alpha\beta}(x)\psi_{\beta,x}(p), \tag{1}$$

for $h_{\alpha\beta}(x) \in \mathrm{Aut}(C\ell_{1,3})$, where $h_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{Aut}(C\ell_{1,3})$ are transition mappings. As every automorphism of $C\ell_{1,3}$ is inner, thus

$$h_{\alpha\beta}(x)\psi_{\beta,x}(p) = a_{\alpha\beta}(x)\psi_{\alpha,x}(p)a_{\alpha\beta}(x)^{-1},$$

where $a_{\alpha\beta}(x) \in C\ell_{1,3}$ is invertible.

(b) The group $\mathrm{SO}^e_{1,3}$ is extended in the Clifford algebra $C\ell_{1,3}$: in fact, as the group $C\ell^*_{1,3} \subset C\ell_{1,3}$ of invertible elements acts on $C\ell_{1,3}$ as an algebra automorphism through by its adjoint representation, a set of *lifts* of the transition functions of $C\ell(M,g)$ is constituted by elements $\{a_{\alpha\beta}\} \subset C\ell^*_{1,3}$ such that if $\mathrm{ad}_\phi(\tau) = \phi\tau\phi^{-1}$, for all $\tau \in C\ell_{1,3}$, hence $\mathrm{ad}_{a_{\alpha\beta}} = h_{\alpha\beta}$.

(c) As $\mathrm{Spin}^e_{1,3} = \{\phi \in C\ell^0_{1,3} \,|\, \phi\tilde{\phi} = 1\} \simeq \mathrm{SL}(2,\mathbb{C})$ is the universal covering group for $\mathrm{SO}^e_{1,3}$, accordingly $\sigma = \mathrm{Ad}|_{\mathrm{Spin}^e_{1,3}}$ defines a group homeomorphism $\sigma : \mathrm{Spin}^e_{1,3} \to \mathrm{SO}^e_{1,3}$ which is onto and has kernel $\mathbb{Z}_2$. Since $\mathrm{Ad}_{-1}$ equals the identity map, thus $\mathrm{Ad} : \mathrm{Spin}^e_{1,3} \to \mathrm{Aut}(C\ell_{1,3})$ descends to a representation of $\mathrm{SO}^e_{1,3}$. One denominates $\mathrm{ad}'$ this representation, namely $\mathrm{ad}' : \mathrm{SO}^e_{1,3} \to \mathrm{Aut}(C\ell_{1,3})$. Thereon one denotes $\mathrm{ad}'_{\sigma(\phi)}\omega = \mathrm{ad}_\phi\omega = \phi\omega\phi^{-1}$.

(d) The group structure associated to the Clifford bundle $C\ell(M,g)$ is reducible from $\mathrm{Aut}(C\ell_{1,3})$ to $\mathrm{SO}^e_{1,3}$. The transition maps of the principal bundle of oriented Lorentz cotetrads $P_{\mathrm{SO}^e_{1,3}}(M)$ are regarded transition maps for the Clifford bundle. Thereupon it follows that $C\ell(M,g) = P_{\mathrm{SO}^e_{1,3}}(M) \times_{\mathrm{Ad}'} C\ell_{1,3}$, meaning that the Clifford bundle is a vector bundle associated to the principal bundle $P_{\mathrm{SO}^e_{1,3}}(M)$ of orthonormal Lorentz coframes. In a spin manifold we have $C\ell(M,g) = P_{\mathrm{Spin}^e_{1,3}}(M) \times_{\mathrm{Ad}} C\ell_{1,3}$. Consequently, spinor fields are sections of vector bundles associated with the principal bundle related to spinor coframes. Dirac spinor fields are sections of the bundle $S(M,g) = P_{\mathrm{Spin}^e_{1,3}}(M) \times_\rho \mathbb{C}^4$, with $\rho$ the $D^{(1/2,0)} \oplus D^{(0,1/2)}$ representation of $\mathrm{Spin}^e_{1,3} \cong \mathrm{SL}(2,\mathbb{C})$ in the space of endomorphisms $\mathrm{End}(\mathbb{C}^4)$.

## 3. Bilinear covariants

This section is devoted to recalling the bilinear covariants. In this article all spinor fields live in the a 4-dimensional spacetime $(M, \eta, D, \tau_\eta, \uparrow)$ which locally has the Lorentzian metric $\eta(\partial/\partial x^\mu, \partial/\partial x^\nu) = \eta_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$. Hence forward $\{x^\mu\}$ are global coordinates adapted to an inertial reference frame $e_0 = \partial/\partial x^0$, and $e_i = \partial/\partial x^i$, $i = 1,2,3$. Moreover, the set $\{e_\mu\}$ is constituted by sections of the frame bundle $\mathbf{P}_{\mathrm{SO}^e_{1,3}}(M)$, related to a set of reciprocal frames $\{\mathbf{e}^\mu\}$ satisfying $\eta(\mathbf{e}^\mu, e_\nu) := \mathbf{e}^\mu \cdot e_\nu = \delta^\mu_\nu$. Let $\{\theta^\mu\}$ $[\{\theta_\mu\}]$ be bases dual to $\{e_\mu\}$ $[\{\mathbf{e}^\mu\}]$. Classical spinor fields are elements of the carrier space associated to a $D^{(1/2,0)} \oplus D^{(0,1/2)}$ representation of $\mathrm{SL}(2,\mathbb{C})$, namely, sections of the vector bundle $\mathbf{P}_{\mathrm{Spin}^e_{1,3}}(M) \times_\rho \mathbb{C}^4$, where $\rho$ stands for the above mentioned $D^{(1/2,0)} \oplus D^{(0,1/2)}$ representation. In addition, the classical spinor fields carrying the $D^{(1/2,0)}$ or the $D^{(0,1/2)}$) representation of $\mathrm{SL}(2,\mathbb{C})$ are sections in the vector bundle $\mathbf{P}_{\mathrm{Spin}^e_{1,3}}(M) \times_{\rho'} \mathbb{C}^2$, where $\rho'$ stands for the $D^{(1/2,0)}$ or the $D^{(0,1/2)}$ representation of $\mathrm{SL}(2,\mathbb{C})$ in $\mathbb{C}^2$. Given a spinor field $\psi \in \sec \mathbf{P}_{\mathrm{Spin}^e_{1,3}}(M) \times_\rho \mathbb{C}^4$, the bilinear covariants may be taken as the following sections of the exterior algebra bundle $\bigwedge TM$:

$$\sigma = \psi^\dagger\gamma_0\psi, \quad \mathbf{J} = J_\mu\theta^\mu = \psi^\dagger\gamma_0\gamma_\mu\psi\theta^\mu, \quad \mathbf{S} = S_{\mu\nu}\theta^{\mu\nu} = \tfrac{1}{2}\psi^\dagger\gamma_0 i\gamma_{\mu\nu}\psi\theta^\mu \wedge \theta^\nu,$$
$$\mathbf{K} = K_\mu\theta^\mu = \psi^\dagger\gamma_0 i\gamma_{0123}\gamma_\mu\psi\theta^\mu, \quad \omega = -\psi^\dagger\gamma_0\gamma_{0123}\psi. \tag{2}$$

Furthermore, the set $\{\mathbf{1}_4, \gamma_\mu, \gamma_\mu\gamma_\nu, \gamma_\mu\gamma_\nu\gamma_\rho, \gamma_0\gamma_1\gamma_2\gamma_3\}$ ($\mu, \nu, \rho = 0,1,2,3$, and $\mu < \nu < \rho$) is a basis for $\mathcal{M}(4,\mathbb{C})$ satisfying [13] $\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2\eta_{\mu\nu}\mathbf{1}_4$ and the Clifford product is denoted by juxtaposition [11, 12].

Concerning the electron, described by Dirac spinor fields (classes 1, 2 and 3 below), $\mathbf{J}$ is a timelike vector corresponding to the current of probability. The bivector $\mathbf{S}$ represents the intrinsic angular momentum distribution, and the spacelike vector $\mathbf{K}$ provides the direction of the electron spin. The bilinear covariants satisfy the Fierz identities [13–15]

$$\mathbf{J}^2 = \omega^2 + \sigma^2, \quad \mathbf{K}^2 = -\mathbf{J}^2, \quad \mathbf{J} \llcorner \mathbf{K} = 0, \quad \mathbf{J} \wedge \mathbf{K} = -(\omega + \sigma\gamma_{0123})\mathbf{S}. \tag{3}$$

When $\omega = 0 = \sigma$, a spinor field is said to be *singular*.

Lounesto [13] has classified spinor fields into the following six disjoint classes. In the classes (1), (2), and (3) below it is implicit that $\mathbf{J}$, $\mathbf{K}$ and $\mathbf{S}$ are all nonzero:

1) $\sigma \neq 0, \quad \omega \neq 0.$
2) $\sigma \neq 0, \quad \omega = 0.$
3) $\sigma = 0, \quad \omega \neq 0.$
4) $\sigma = 0 = \omega, \quad \mathbf{K} \neq 0, \quad \mathbf{S} \neq 0.$
5) $\sigma = 0 = \omega, \quad \mathbf{K} = 0, \quad \mathbf{S} \neq 0.$
6) $\sigma = 0 = \omega, \quad \mathbf{K} \neq 0, \quad \mathbf{S} = 0.$

Spinor fields of types (1), (2), and (3) are called *Dirac spinor fields* for spin-1/2 particles while spinor fields of types (4), (5), and (6) are called, respectively, *flag-dipoles*, *flagpoles* and Weyl spinor fields. Despite $\mathbf{J} \neq 0$, for these three types the vectors $\mathbf{J}$ and $\mathbf{K}$ are always timelike. Furthermore, a complex multivector field can be introduces as [13]

$$Z = \sigma + \mathbf{J} + i\mathbf{S} + i\mathbf{K}\gamma_{0123} + \omega\gamma_{0123}, \tag{4}$$

where the multivector operators $\sigma$, $\omega$, $\mathbf{J}$, $\mathbf{S}$ and $\mathbf{K}$ satisfy the Fierz identities. It is denominated the *Fierz aggregate*. Moreover, if $\gamma_0 Z^\dagger\gamma_0 = Z$, $Z$ is called a *boomerang*. With respect to a singular spinor field ($\omega = 0 = \sigma$), the Fierz identities are replaced by the more general conditions [14]

$$Z\gamma_\mu Z = 4J_\mu Z, \qquad Z^2 = 4\sigma Z, \qquad Zi\gamma_{\mu\nu}Z = 4S_{\mu\nu}Z,$$
$$Z\gamma_{0123}Z = -4\omega Z, \qquad Zi\gamma_{0123}\gamma_\mu Z = 4K_\mu Z. \tag{5}$$

## 4. Classical spinors, algebraic spinors and spinor operators

Given an orthonormal basis $\{e_\mu\}$ in $\mathbb{R}^{1,3}$, an arbitrary element of $C\ell_{1,3}$ is written as

$$\Psi = b + b^\mu e_\mu + b^{\mu\nu} e_{\mu\nu} + b^{\mu\nu\sigma} e_{\mu\nu\sigma} + p e_{0123}, \tag{6}$$

From the isomorphism $C\ell_{1,3} \simeq \mathcal{M}(2,\mathbb{H})$, in order to obtain a representation of $C\ell_{1,3}$, a primitive idempotent $f = \frac{1}{2}(1+e_0)$ is used. An arbitrary element of the left minimal ideal $C\ell_{1,3}f$ is given by

$$\Omega = (a^1 + a^2 e_{23} + a^3 e_{31} + a^4 e_{12}) f + (a^5 + a^6 e_{23} + a^7 e_{31} + a^8 e_{12}) e_{0123} f,$$

If we set $\Omega = \Psi f \in C\ell_{1,3}f$, then

$$a^1 = b + b^0, \quad a^2 = b^{23} + b^{023}, \quad a^3 = -b^{13} - b^{013}, \quad a^4 = b^{12} + b^{012},$$

$$a^5 = -b^{123} + p, \quad a^6 = b^1 - b^{01}, \quad a^7 = b^2 - b^{02}, \quad a^8 = b^3 - b^{03}.$$

In fact, although $C\ell_{1,3}f$ is a minimal left ideal, it is a right $\mathbb{H}$-module, therefore the quaternionic coefficients should be written to the right of $f$. Let $q_1$ and $q_2$ be two quaternions

$$q_1 = a^1 + a^2 e_{23} + a^3 e_{31} + a^4 e_{12}, \quad q_2 = a^5 + a^6 e_{23} + a^7 e_{31} + a^8 e_{12} \in \mathbb{H} \tag{7}$$

where $\mathbb{K} = fC\ell_{1,3}f = \mathrm{span}_{\mathbb{R}}\{1, e_{23}, e_{31}, e_{12}\} \cong \mathbb{H}$. The quaternionic coefficients $q_1, q_2$ commute with $f$ and with $e_{0123}$ thus $fq_1 + e_{0123} f q_2 = q_1 f + q_2 e_{0123} f$. The left ideal $C\ell_{1,3}f$ is a right module over $\mathbb{K}$ and as such its basis is $\{f, e_{0123}f\}$. By denoting $\mathfrak{i} = e_{23}$, $\mathfrak{j} = e_{31}$, and $\mathfrak{k} = e_{12}$, in the representation

$$e_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \; e_1 = \begin{pmatrix} 0 & \mathfrak{i} \\ \mathfrak{i} & 0 \end{pmatrix}, \; e_2 = \begin{pmatrix} 0 & \mathfrak{j} \\ \mathfrak{j} & 0 \end{pmatrix}, \; e_3 = \begin{pmatrix} 0 & \mathfrak{k} \\ \mathfrak{k} & 0 \end{pmatrix}, \tag{8}$$

the elements $f$ and $e_{0123}f$ are represented as $[f] = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $[e_{0123}f] = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. Thus, any element $\Psi \in C\ell_{1,3}$ can be represented as the following quaternionic matrix:

$$\begin{pmatrix} \begin{matrix} b + b^0 + (b^{23} + b^{023})\mathfrak{i} \\ -(b^{13} + b^{013})\mathfrak{j} + (b^{12} + b^{012})\mathfrak{k} \end{matrix} & \begin{matrix} -b^{123} - p + (b^1 + b^{01})\mathfrak{i} + \\ (b^2 + b^{02})\mathfrak{j} + (b^3 + b^{03})\mathfrak{k} \end{matrix} \\ \\ \begin{matrix} p - b^{123} + (b^1 - b^{01})\mathfrak{i} \\ +(b^2 - b^{02})\mathfrak{j} + (b^3 - b^{03})\mathfrak{k} \end{matrix} & \begin{matrix} b - b^0 + (b^{23} - b^{023})\mathfrak{i} + \\ (b^{013} - b^{13})\mathfrak{j} + (b^{12} - b^{012})\mathfrak{k} \end{matrix} \end{pmatrix} = \begin{pmatrix} q_1 & q_3 \\ q_2 & q_4 \end{pmatrix}. \tag{9}$$

Spinor fields were constructed by differential forms by Fock, Ivanenko, and Landau in 1928 and also in [13]. A spinor operator $\mathring{\Psi} \in C\ell_{1,3}^+$ is written as

$$\mathring{\Psi} = b + b^{\mu\nu} e_{\mu\nu} + p e_{0123}, \tag{10}$$

which in the light of (9) reads

$$\mathring{\Psi} = \begin{pmatrix} b + b^{23}\mathfrak{i} - b^{13}\mathfrak{j} + b^{12}\mathfrak{k} & -b^{0123} + b^{01}\mathfrak{i} + b^{02}\mathfrak{j} + b^{03}\mathfrak{k} \\ b^{0123} - b^{01}\mathfrak{i} - b^{02}\mathfrak{j} - b^{03}\mathfrak{k} & b + b^{23}\mathfrak{i} - b^{13}\mathfrak{j} + b^{12}\mathfrak{k} \end{pmatrix} = \begin{pmatrix} q_1 & -q_2 \\ q_2 & q_1 \end{pmatrix}$$

where $b^{0123} = p$. Now, the vector space isomorphisms

$$C\ell_{1,3}^{+} \simeq C\ell_{3,0} \simeq C\ell_{1,3}\tfrac{1}{2}(1+e_0) \simeq \mathbb{C}^4 \simeq \mathbb{H}^2$$

give the equivalence among the classical, the operatorial, and the algebraic definitions of a spinor. In this sense, the spinor space $\mathbb{H}^2$ which carries the $D^{(1/2,0)} \oplus D^{(0,1/2)}$ or $D^{(1/2,0)}$, or $D^{(0,1/2)}$ representations of $\mathrm{SL}(2,\mathbb{C})$ is isomorphic to the minimal left ideal $C\ell_{1,3}\frac{1}{2}(1+e_0)$ – corresponding to the algebraic spinor – and also isomorphic to the even subalgebra $C\ell_{1,3}^{+}$ – corresponding to the operatorial spinor. It is hence possible to write a Dirac spinor field as

$$\begin{aligned}\begin{pmatrix} q_1 & -q_2 \\ q_2 & q_1 \end{pmatrix}[f] &= \begin{pmatrix} q_1 & -q_2 \\ q_2 & q_1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} q_1 & 0 \\ q_2 & 0 \end{pmatrix} \\ &\simeq \begin{pmatrix} b + b^{23}\mathfrak{i} - b^{13}\mathfrak{j} + b^{12}\mathfrak{k} \\ p - b^{01}\mathfrak{i} - b^{02}\mathfrak{j} - b^{03}\mathfrak{k} \end{pmatrix} \in C\ell_{1,3}f \simeq \mathbb{H}\oplus\mathbb{H}. \end{aligned} \tag{11}$$

Returning to (10), and using for instance the standard representation,

$$1 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathfrak{i} \mapsto \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \mathfrak{j} \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \mathfrak{k} \mapsto \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \tag{12}$$

the complex matrix associated to the spinor operator $\mathring{\Psi}$ looks as follows:

$$\begin{aligned}[\mathring{\Psi}] &= \begin{pmatrix} b + b^{23}i & -b^{13} + b^{12}i & -b^{0123} + b^{01}i & b^{02} + b^{03}i \\ b^{13} + b^{12}i & b - b^{23}i & -b^{02} + b^{03}i & -p - b^{01}i \\ b^{0123} - b^{01}i & -b^{02} - b^{03}i & b + b^{23}i & -b^{13} + b^{12}i \\ b^{02} - b^{03}i & p + b^{01}i & b^{13} + b^{12}i & b - b^{23}i \end{pmatrix} \\ &:= \begin{pmatrix} \phi_1 & -\phi_2^* & -\phi_3 & \phi_4^* \\ \phi_2 & \phi_1^* & -\phi_4 & -\phi_3^* \\ \phi_3 & -\phi_4^* & \phi_1 & -\phi_2^* \\ \phi_4 & \phi_3^* & \phi_2 & \phi_1^* \end{pmatrix}. \end{aligned}$$

The Dirac spinor $\psi$ is an element of the minimal left ideal $(\mathbb{C}\otimes C\ell_{1,3})f$ where

$$f = \tfrac{1}{4}(1+e_0)(1+ie_{12}) \tag{13}$$

is a primitive idempotent that gives the Dirac representation (instead, one could set

$$f = \tfrac{1}{4}(1+ie_{0123})(1+ie_{12}), \tag{14}$$

that would give the Weyl representation). In this standard Dirac representation, the basis vectors $e_\mu$ are sent to $\gamma_\mu \in \mathrm{End}(\mathbb{C}^4)$ (see (A.3) in Appendix A). Therefore,

$$\psi = \Phi\tfrac{1}{2}(1+i\gamma_{12}) \in (\mathbb{C}\otimes C\ell_{1,3})f$$

where $\Phi = \Phi\frac{1}{2}(1+\gamma_0) \in C\ell_{1,3}(1+\gamma_0)$ is twice the real part of $\psi$ [13, Sec. 10.3]. Using this representation it follows that

$$\psi \simeq \begin{pmatrix} \phi_1 & 0 & 0 & 0 \\ \phi_2 & 0 & 0 & 0 \\ \phi_3 & 0 & 0 & 0 \\ \phi_4 & 0 & 0 & 0 \end{pmatrix} \in (\mathbb{C}\otimes C\ell_{1,3})f \simeq \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{pmatrix} \in \mathbb{C}^4. \tag{15}$$

Thus, the above allows for identifying the algebraic spinor fields with the classical Dirac spinor fields.

## 5. Quantum algebraic spinor fields

An arbitrary bilinear form $B : V \times V \to \mathbb{R}$ can be written as $B = g + A$, where $g = \frac{1}{2}(B + B^T)$ and $A = \frac{1}{2}(B - B^T)$. Specifically, we let

$$g = \mathrm{diag}(1, -1, -1, -1) \quad \text{and} \quad A = \begin{pmatrix} 0 & A_{01} & A_{02} & A_{03} \\ -A_{01} & 0 & A_{12} & A_{13} \\ -A_{02} & -A_{12} & 0 & A_{23} \\ -A_{03} & -A_{13} & -A_{23} & 0 \end{pmatrix}.$$

Let $u, v, w \in V$. The form defines an annihilation operator $I_u(v) := u \underset{B}{\lrcorner} v = B(u, v)$ which is extended to $\bigwedge V$. Given $\phi, \psi \in \bigwedge V$, the annihilation and creation operators $I_u, E_u : \bigwedge V \to \bigwedge V$ are respectively defined as

$$\begin{aligned} &I_u(\phi \wedge \psi) = (u \underset{B}{\lrcorner} \phi) \wedge \psi + \hat{\phi} \wedge (u \underset{B}{\lrcorner} \psi), \\ &E_u(\psi) = u \wedge \psi \end{aligned} \tag{16}$$

where $\hat{\phi}$ denotes the grade involution of $\phi$. The maps in (16) induce a Clifford map $\Gamma_u = E_u + I_u : \bigwedge V \to \bigwedge V$ satisfying

$$\begin{aligned} &\Gamma_u \circ \Gamma_v + \Gamma_v \circ \Gamma_u = 2g(u, v), \\ &\Gamma_u \circ \Gamma_v - \Gamma_v \circ \Gamma_u = 2(\Gamma_{u \wedge v} + A(u, v)), \end{aligned} \tag{17}$$

where $A(u, v) = u \underset{A}{\lrcorner} v$. Since $\Gamma_u \circ \Gamma_v = \Gamma_{u\wedge v} + B(u, v)$, namely $(uv)_B = u \wedge v + B(u, v)$, the equations above can be written as

$$(uv)_B + (vu)_B = 2g(u, v)1, \qquad (uv)_B - (vu)_B = 2(u \wedge v + A(u, v)), \tag{18}$$

showing that the antisymmetric part $A$ defines another $\mathbb{Z}_n$-grading. Here, $(uv)_B$ denotes the Clifford product between $u$ and $v$ in $C\ell(V, B)$. Indeed,

$$(uv)_B := uv + A(u, v)1 = u \wedge v + g(u, v) + A(u, v) = u \wedge v + B(u, v),$$

where $uv$ denotes the Clifford product between $u$ and $v$ in $C\ell(V, g)$. Thus, there is another exterior product – the *dotted wedge* [13] – $\dot{\wedge}$ induced by $A$: $u \dot{\wedge} v = u \wedge v + A(u, v)$. In general, the $A$-induced $\mathbb{Z}_n$ grading is given by $\dot{\bigwedge}^{2k} V = \bigwedge^0 V \oplus \cdots \oplus \bigwedge^{2k} V$ and $\dot{\bigwedge}^{2k+1} V = \bigwedge^0 V \oplus \cdots \oplus \bigwedge^{2k+1} V$.

Given $u, v, w \in V$ and $\psi \in C\ell(V, g)$, the $B$-products – namely, the Clifford product induced by the arbitrary bilinear form $B$ – between one, two, and three vectors and arbitrary multivectors are provided, respectively, as follows:

$$\begin{aligned} &(u\psi)_B = u\psi + u \underset{A}{\lrcorner} \psi, \\ &[(uv)\psi]_B = uv\psi + u(v \underset{A}{\lrcorner} \psi) - v \wedge (u \underset{A}{\lrcorner} \psi) + u \underset{A}{\lrcorner} (v \underset{B}{\lrcorner} \psi), \end{aligned}$$

$$
\begin{aligned}
[(uvw)\psi]_B = {} & uvw\psi + uv(w \underset{A}{\lrcorner} \psi) - uw(u \underset{g}{\lrcorner} \psi) + w \wedge (u \underset{g}{\lrcorner} (v \underset{A}{\lrcorner} \psi)) + u(v \underset{A}{\lrcorner} (w \underset{B}{\lrcorner} \psi)) \\
& - v \wedge (u \underset{A}{\lrcorner} w)\psi + v \wedge w \wedge (u \underset{A}{\lrcorner} \psi) - v \wedge (u \underset{g-A}{\lrcorner} (w \underset{g}{\lrcorner} \psi)) + u \underset{A}{\lrcorner} ((v \underset{g}{\lrcorner} w)\psi) \\
& - v \wedge (u \underset{A}{\lrcorner} (w \underset{B}{\lrcorner} \psi)) - (w \underset{A}{\lrcorner} u)v\psi - (v \underset{A}{\lrcorner} w)u\psi. \qquad (19)
\end{aligned}
$$

In (15) we used the minimal ideal provided by the idempotent

$$f = \tfrac{1}{4}(1 + \gamma_0)(1 + i\gamma_1\gamma_2) = \tfrac{1}{4}(1 + \gamma_0 + i\gamma_1\gamma_2 + i\gamma_0\gamma_1\gamma_2).$$

Now, in $C\ell(V, B)$ the formalism is recovered when we consider the idempotent

$$f_B = \tfrac{1}{4}(1 + \gamma_0 + i\gamma_1 {}_B\gamma_2 + i\gamma_0 {}_B\gamma_1 {}_B\gamma_2) \qquad (20)$$

where we let $\gamma_1 {}_B\gamma_2 = (\gamma_1\gamma_2)_B$, $\gamma_0 {}_B\gamma_1 {}_B\gamma_2 = (\gamma_0\gamma_1\gamma_2)_B$, etc. in $C\ell(V, B)$. The formalism for $C\ell(V, B)$ is *mutatis mutandis* obtained, just by changing the standard Clifford product $\gamma_\mu\gamma_\nu$ to

$$\gamma_\mu {}_B\gamma_\nu = \gamma_\mu\gamma_\nu + A_{\mu\nu} \qquad (21)$$

The last expression is the prominent essence of transliterating $C\ell(V, B)$ to $C\ell(V, g)$. For instance, (15) evinces the necessity of defining

$$f = \tfrac{1}{4}(1 + \gamma_0)(1 + i\gamma_1\gamma_2) \in C\ell(V, g). \qquad (22)$$

Now, in $\mathbb{C} \otimes C\ell^B_{1,3}$ we have

$$
\begin{aligned}
f_B &= \tfrac{1}{4}(1 + \gamma_0) {}_B (1 + i\gamma_1 {}_B\gamma_2) \\
&= \tfrac{1}{4}(1 + \gamma_0)(1 + i\gamma_1\gamma_2) + \tfrac{i}{4}(A_{12} + A_{12}\gamma_0 - A_{02}\gamma_1 + A_{01}\gamma_2). \qquad (23)
\end{aligned}
$$

Herein we shall denote

$$f_B = f + f(A) \qquad (24)$$

where $f(A) = \frac{i}{4}(A_{12} + A_{12}\gamma_0 - A_{02}\gamma_1 + A_{01}\gamma_2)$.

In the Dirac representation (A.3), the idempotent $f$ in (22) reads

$$f = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and as

$$\gamma_0 {}_B\gamma_1 {}_B\gamma_2 = \gamma_0\gamma_1\gamma_2 + A_{01}\gamma_2 - A_{02}\gamma_1 + A_{12}\gamma_0, \qquad (25)$$

one can substitute it in (24) to obtain

$$
\begin{aligned}
f_B = {} & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
& + \frac{1}{4} \begin{pmatrix} 2iA_{12} & 0 & 0 & -iA_{02} - A_{01} \\ 0 & 2iA_{12} & -iA_{02} + A_{01} & 0 \\ 0 & iA_{02} + A_{01} & 0 & 0 \\ iA_{02} - A_{01} & 0 & 0 & 0 \end{pmatrix}. \qquad (26)
\end{aligned}
$$

When $A_{\mu\nu} = 0$ it implies that $B = g$ and the standard spinor formalism is recovered. Let us denote by $C\ell^B_{1,3}$ the Clifford algebra $C\ell(V, B)$, where $V = \mathbb{R}^4$ and $B = \eta + A$, where $\eta$ denotes the Minkowski metric.

An arbitrary element of $C\ell^B_{1,3}$ is written as

$$\psi_B = c + c^\mu\gamma_\mu + c^{\mu\nu}(\gamma_\mu\gamma_\nu)_B + c^{\mu\nu\sigma}(\gamma_\mu\gamma_\nu\gamma_\sigma)_B + p(\gamma_0\gamma_1\gamma_2\gamma_3)_B. \tag{27}$$

By using (21, 25), (27) reads

$$\psi_B = \psi + c^{\mu\nu}A_{\mu\nu} + c^{\mu\nu\rho}(A_{\mu\nu}\gamma_\rho + A_{\rho\mu}\gamma_\nu + A_{\nu\rho}\gamma_\mu) + p\epsilon^{\mu\nu\rho\sigma}A_{\mu\nu}(\gamma_{\rho\sigma} + A_{\rho\sigma}) \tag{28}$$

where $\psi$ is an element in the standard Clifford algebra $C\ell_{1,3}$ of the form given by (6).

Herein we shall rewrite (28) as

$$\psi_B = \psi + \psi(A) \tag{29}$$

where it indicates that an arbitrary element of $C\ell^B_{1,3}$ is written as the sum of an arbitrary element of $C\ell_{1,3}$ and an $A$-dependent element of $C\ell_{1,3}$, where $\psi(A) \in \bigwedge^0 V \oplus \bigwedge^1 V \oplus \bigwedge^2 V$. Furthermore, we denoted above

$$\psi(A) = (c^{\mu\nu}A_{\mu\nu} + p\epsilon^{\mu\nu\rho\sigma}A_{\mu\nu}A_{\rho\sigma}) + c^{\mu\nu\rho}(A_{\mu\nu}\gamma_\rho + A_{\rho\mu}\gamma_\nu + A_{\nu\rho}\gamma_\mu) + p\epsilon^{\mu\nu\rho\sigma}A_{\mu\nu}\gamma_{\rho\sigma} \tag{30}$$

where $\epsilon^{\mu\nu\rho\sigma}$ denotes the Levi-Civita symbol.

An algebraic $B$-spinor is defined to be an element $(\psi_B)\,{}_{\underset{B}{}}\,(f_B)$ of a minimal left ideal $(\mathbb{C} \otimes C\ell^B_{1,3})f_B$ generated by a primitive idempotent $f_B$ in $\mathbb{C} \otimes C\ell^B_{1,3}$. In Appendix A, we discuss primitive idempotents in $\mathbb{C} \otimes C\ell^B_{1,3}$ which give an orthogonal decomposition of the unity and we calculate a basis for the ideal $(\mathbb{C} \otimes C\ell^B_{1,3})f_B$. Now, by using (24) and (29), and remembering that the usual Clifford product in $C\ell_{1,3}$ is denoted by juxtaposition, any algebraic spinor can be written as

$$\begin{aligned}(\psi_B)\underset{B}{}(f_B) &= (\psi + \psi(A))\underset{B}{}(f + f(A)) \\ &= \psi\underset{B}{}f + \psi(A)\underset{B}{}f + \psi\underset{B}{}f(A) + \psi(A)\underset{B}{}f(A)\end{aligned} \tag{31}$$

Here upon we shall denote by $s$ the scalar part $c^{\mu\nu}A_{\mu\nu} + p\epsilon^{\mu\nu\rho\sigma}A_{\mu\nu}A_{\rho\sigma}$ of $\psi(A)$ in (30). Each term in (31) is explicitly calculated in Appendix B.

## 6. Spinor field classification in quantum Clifford algebras

In order to provide physical insight into the mathematical formalism presented in the context of the spinor fields classification, the formalism presented in Section 3 is now analyzed in the context of the quantum Clifford algebra $C\ell(V, B)$. In particular, we aim to describe the correspondence between spinor fields in $C\ell(V, g)$ and the quantum spinor fields, or the *B-spinor fields*, in $C\ell(V, B)$ where $V$ denotes the 4-dimensional Minkowski spacetime.

As in the orthogonal Clifford algebraic formalism, the quantum spinor fields classification is provided by the following spinor field classes:

$1_B)$ $\sigma_B \neq 0, \quad \omega_B \neq 0.$

$2_B$) $\sigma_B \neq 0, \quad \omega_B = 0.$

$3_B$) $\sigma_B = 0, \quad \omega_B \neq 0.$

$4_B$) $\sigma_B = 0 = \omega_B, \quad \mathbf{K}_B \neq 0, \quad \mathbf{S}_B \neq 0.$

$5_B$) $\sigma_B = 0 = \omega_B, \quad \mathbf{K}_B = 0, \quad \mathbf{S}_B \neq 0.$

$6_B$) $\sigma_B = 0 = \omega_B, \quad \mathbf{K}_B \neq 0, \quad \mathbf{S}_B = 0.$

It is always possible to write:

$$\sigma_B = \sigma + \sigma(A), \tag{32}$$

$$\mathbf{J}_B = \mathbf{J} + \mathbf{J}(A), \tag{33}$$

$$\mathbf{S}_B = \mathbf{S} + \mathbf{S}(A), \tag{34}$$

$$\mathbf{K}_B = \mathbf{K} + \mathbf{K}(A), \tag{35}$$

$$\omega_B = \omega + \omega(A). \tag{36}$$

In general, since we assume $A \neq 0$ (otherwise there is nothing new to prove, as when $A = 0$ it implies that $C\ell(V, B) = C\ell(V, g)$), it follows that all the $A$-dependent quantities $\sigma(A)$, $\mathbf{J}(A)$, $\mathbf{S}(A)$, $\mathbf{K}(A)$, and $\omega_A$ do not equal zero. The expressions for such $A$-independent terms are developed in the Appendix. There is an immediate correspondence between the spinor fields in the Lounesto classification and the quantum spinor fields that are distributed in the six classes $1_B) - 6_B)$ above. More precisely, all possibilities are analyzed in what follows:

$1_B$) $\sigma_B \neq 0$, $\omega_B \neq 0$. As $\sigma_B \neq 0$ and $\sigma_B = \sigma + \sigma(A)$, we have some possibilities, depending whether $\sigma$ does or does not equal zero, as well as $\omega$:

*i*) $\sigma = 0 = \omega$. This case corresponds to the type-(4), type-(5), and type-(6) spinor fields – respectively flag-dipoles, flagpoles, and Weyl ones. Such possibility is obviously compatible to $\sigma_B \neq 0, \quad \omega_B \neq 0$.

*ii*) $\sigma = 0$ and $\omega \neq 0$. This case corresponds to the type-(3) Dirac spinor fields. The condition $\sigma = 0$ is compatible to $\sigma_B \neq 0$, but as $\omega \neq 0$, the additional condition $\omega_B = \omega + \omega(A) \neq 0$ must be imposed. Equivalently, $0 \neq \omega \neq \omega(A)$.

*iii*) $\sigma \neq 0$ and $\omega = 0$. This case corresponds to the type-(2) Dirac spinor fields. The condition $\omega = 0$ is compatible to $\omega_B \neq 0$, but as $\sigma \neq 0$, the additional condition $\sigma_B = \sigma + \sigma(A) \neq 0$ is demanded. Equivalently, $0 \neq \sigma \neq \sigma(A)$.

*iv*) $\sigma \neq 0$ and $\omega \neq 0$. This case corresponds to the type-(1) Dirac spinor fields. Here both the conditions $0 \neq \omega \neq \omega(A)$ and $0 \neq \sigma \neq \sigma(A)$ must be imposed.

All the conditions heretofore must hold in order so that the $B$-spinor field be a representing spinor field in class $1_B$).

$2_B$) $\sigma_B \neq 0$, $\omega_B = 0$. Although the condition $\sigma_B \neq 0$ is compatible to the possibilities $\sigma = 0$ and $\sigma \neq 0$ as well (clearly the condition $\sigma \neq 0$ is compatible to $\sigma_B \neq 0$ if $\sigma \neq -\sigma(A)$), the condition $\omega_B = 0$ implies that $\omega = -\omega(A)$, which does not equal zero. To summarize:

*i*) $\sigma = 0$ and $\omega \neq 0$. This case corresponds to the type-(3) Dirac spinor fields. The condition $\sigma = 0$ is compatible to $\sigma_B \neq 0$, but as $\omega \neq 0$, the additional

**Table 1.** Correspondence among the spinor field and the (quantum) $B$-spinor fields under Lounesto spinor field classification.

| | Quantum Spinor Fields | Spinor Fields | |
|---|---|---|---|
| type-($1_B$) | $B$-Dirac | Dirac | type-(1) |
| | | Dirac | type-(2) |
| | | Dirac | type-(3) |
| | | Flag-dipoles | type-(4) |
| | | Flagpoles (also Elko, Majorana) | type-(5) |
| | | Weyl | type-(6) |
| type-($2_B$) | $B$-Dirac | Dirac | type-(3) |
| | | Dirac | type-(1) |
| type-($3_B$) | $B$-Dirac | Dirac | type-(2) |
| | | Dirac | type-(1) |
| type-($4_B$) | $B$-flag-dipole | Dirac | type-(1) |
| type-($5_B$) | $B$-flagpole | Dirac | type-(1) |
| type-($6_B$) | $B$-Weyl | Dirac | type-(1) |

condition $\omega_B = \omega + \omega(A) \neq 0$ must hold. It is tantamount to assert that $0 \neq \omega \neq \omega(A)$.

*ii*) $\sigma \neq 0$ and $\omega \neq 0$. This case corresponds to the type-(1) Dirac spinor fields. Here both the conditions $0 \neq \omega \neq \omega(A)$ and $0 \neq \sigma \neq \sigma(A)$ has to hold.

$3_B$) $\sigma_B = 0, \omega_B \neq 0$. Despite the condition $\omega_B \neq 0$ is compatible to both the possibilities $\omega = 0$ and $\omega \neq 0$ (clearly the condition $\omega \neq 0$ is compatible to $\omega_B \neq 0$ if $\omega \neq -\omega(A)$), the condition $\sigma_B = 0$ implies that $\sigma = -\sigma(A)$, which does not equal zero. To summarize:

*i*) $\omega = 0$ and $\sigma \neq 0$. This case corresponds to the type-(2) Dirac spinor fields. The condition $\omega = 0$ is compatible to $\omega_B \neq 0$, but as $\sigma \neq 0$, the additional condition $\sigma_B = \sigma + \sigma(A) \neq 0$ must be imposed. Equivalently, $0 \neq \sigma \neq \sigma(A)$.

*ii*) $\sigma \neq 0$ and $\omega \neq 0$. This case corresponds to the type-(1) Dirac spinor fields. Here both the conditions $0 \neq \omega \neq \omega(A)$ and $0 \neq \sigma \neq \sigma(A)$ must be imposed.

$4_B$) $\sigma_B = 0 = \omega_B, \quad \mathbf{K}_B \neq 0, \quad \mathbf{S}_B \neq 0.$

$5_B$) $\sigma_B = 0 = \omega_B, \quad \mathbf{K}_B = 0, \quad \mathbf{S}_B \neq 0.$

$6_B$) $\sigma_B = 0 = \omega_B, \quad \mathbf{K}_B \neq 0, \quad \mathbf{S}_B = 0.$

All the quantum spinor fields $4_B$), $5_B$), and $6_B$) are defined by the condition $\sigma_B = 0 = \omega_B$. This implies that $\sigma = -\sigma(A)(\neq 0)$, and that $\omega = -\omega(A)(\neq 0)$. It means that all the singular $B$-spinor fields correspond to the type-(1) Dirac spinor fields.

The results regarding the analysis above can be abridged in the table The paramount importance concerning such classification is multifold. For instance, the Dirac spinor fields in quantum Clifford algebras ($B$-Dirac spinor fields) correspond to all types of spinor fields in the standard Lounesto spinor field classification. In particular, type-($1_B$) Dirac spinor fields correspond to all spinor fields in the orthogonal Clifford algebra. A deep discussion about these results is going to be accomplished in the next Section.

## 7. Concluding remarks and outlook

The mathematical apparatus provided by the quantum Clifford algebraic formalism is a powerful tool, in particular to bring additional interpretations about the underlying standard spacetime structures. For instance, equations (32–36) illustrate that the distribution of intrinsic angular momentum, formerly a legitimate bivector in the standard Clifford algebra $C\ell(V,g)$, is now the direct sum of a bivector and a scalar when considered in $C\ell(V,B)$ *from the point of view of* $C\ell(V,g)$, evincing the different $\mathbb{Z}_n$-grading induced by the antisymmetric part of the arbitrary bilinear form $B$. Furthermore, now, the bilinear covariant $\mathbf{K}$ is a paravector – the sum of a vector and a scalar – which is not a homogeneous Clifford element. Indeed, in $C\ell(V,B)$ it is a homogeneous 1-form, but in $C\ell(V,g)$ it is a paravector.

Some questions and possible answers can still be posed in the context of the quantum Clifford algebraic arena. The mathematical formalism concerning quantum Clifford algebras is rich and a plethora of relevant results can be found in the specialized literature from the last decades. Notwithstanding, the physical relevance of such formalism and its applications is a prominent feature to be still explored. The $B$-spinor fields indicated and categorized in Table 1 can probe spacetime attributes.

We already constructed a dynamical transformation that maps types-(1), (2), and (3) Dirac spinor fields into a subset of type-(5) spinor field [17] which is a prime candidate in terms of which one could attempt to describe the dark matter [19,20] and to incorporate the flagpole spinor fields into the Standard Model. Besides, such a mapping revealed to be an instanton Hopf fibration map [16]. Furthermore, the classification shown at Table 1 is an alternative path to encompass the above mentioned mapping between the different spinor field classes. It can be used further to probe the existence of an arbitrary bilinear form $B$ endowing the spacetime structure. Physically, there are some formulations of gravitational theories in spacetimes endowed with arbitrary bilinear forms composed by a (symmetric) metric $g$ and an additional (antisymmetric) part $A$. From the phenomenological point of view, if such antisymmetric part can be detected or probed, it shall be unraveled as a form that has a tiny norm when compared to the norm of the symmetric part $g$, given any norm on the space of bilinear forms.

It is worth to emphasize here that it is not the first time that the Lounesto spinor field classification is employed to probe unexpected spacetime properties. The so called exotic (type-(5)) dark spinor fields have been used to probe topological obstructions is the spacetime structure. The dark spinor fields dynamics imposes constraints in the spacetime metric structure. Meanwhile, such constraints may be alleviated at the cost of constraining the exotic spacetime topology [18]. In addition, the exotic interactions with the Higgs boson can make it phenomenologically explicit that a subset of type-

(5) spinor fields is a prime candidate to describe the dark matter [19,20]. In particular, observational aspects of such a possibility has been proposed at LHC [21]. The formalism presented here can bring some new light on the possibilities of probing the spacetime structure, here provided by the alteration of the spacetime metric structure by the addition of an antisymmetric part to the metric.

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## Appendix A. Primitive idempotents in $\mathbb{C}\otimes C\ell_{1,3}$ and in $\mathbb{C}\otimes C\ell^B_{1,3}$

In this appendix we use notation from Sections IV and V. Recall that the Dirac spinor $\psi$ is an element of the minimal left ideal $S=(\mathbb{C}\otimes C\ell_{1,3})f$ which is generated by a primitive idempotent $f$ shown in (13). Then, as it can be easily checked, either by hand or with `CLIFFORD` [22], a basis for $S$ may be chosen as

$$S=(\mathbb{C}\otimes C\ell_{1,3})f=\mathrm{Span}_{\mathbb{C}}\{f,-e_{13}f,e_{30}f,e_{10}f\}=\mathrm{Span}_{\mathbb{C}}\{f,-e_{13}f,e_3f,e_1f\}. \tag{A.1}$$

The idempotent $f$ is primitive which means it cannot be written as a sum $f=f_1+f_2$ of two non zero mutually annihilating idempotents, that is, satisfying $f_1f_2=f_2f_1=0$. The fact that $f$ is primitive follows from the isomorphism $\mathbb{C}\otimes C\ell_{1,3}\cong C\ell_{2,3}\cong \mathrm{Mat}(4,\mathbb{C})$ and the fact that the Radon-Hurwitz number $r_{q-p}$ for the signature $(2,3)$, i.e., where $p=2,q=3$, is 1. This in turn implies that any idempotent in $C\ell_{2,3}$ of the form

$$f=\tfrac{1}{2}(1\pm\mathbf{e}_{\underline{i}_1})\tfrac{1}{2}(1\pm\mathbf{e}_{\underline{i}_2}), \tag{A.2}$$

where $\mathbf{e}_{\underline{i}_i}$, $i=1,2$, are commuting basis monomials in $C\ell_{2,3}$ with square 1 is a primitive idempotent in $C\ell_{2,3}$ [23]. In fact, the number $k$ of non-primitive idempotent factors in (A.2) equals $k=3-r_{3-2}=2$. Thus, the Dirac standard representation of the algebra $\mathbb{C}\otimes C\ell_{1,3}$ in the ideal $S$ yields the well-known Dirac matrices $\gamma_\mu,\mu=0,1,2,3$:

$$\gamma_0=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix},\quad \gamma_1=\begin{pmatrix}0&0&0&-1\\0&0&-1&0\\0&1&0&0\\1&0&0&0\end{pmatrix},$$

$$\gamma_2=\begin{pmatrix}0&0&0&i\\0&0&-i&0\\0&-i&0&0\\i&0&0&0\end{pmatrix},\quad \gamma_3=\begin{pmatrix}0&0&-1&0\\0&0&0&1\\1&0&0&0\\0&-1&0&0\end{pmatrix} \tag{A.3}$$

which represent, respectively, the algebra generators $e_0,e_1,e_2,e_3$. Since the idempotent $f$ is primitive, the ideal $S$ is minimal, and the representation (A.3) is faithful and irreducible. By alternating signs in the idempotent $f$ shown in (13), we get three additional primitive idempotents. Thus, we get four primitive idempotents $f_1, f_2, f_3, f_4$ which are mutually annihilating and which provide a decomposition of the unity, namely,

$$\begin{aligned} f_1 &= \tfrac{1}{4}(1+e_0)(1+ie_{12}), \quad f_2 = \tfrac{1}{4}(1+e_0)(1-ie_{12}),\\ f_3 &= \tfrac{1}{4}(1-e_0)(1+ie_{12}), \quad f_4 = \tfrac{1}{4}(1-e_0)(1-ie_{12}). \end{aligned} \tag{A.4}$$

where $f_1 = f$, $f_i^2 = f_i$, $f_i f_j = f_j f_i = 0$, for $i, j = 1, \ldots, 4, i \neq j$, and $f_1+f_2+f_3+f_4 = 1$ in $\mathbb{C} \otimes C\ell_{1,3}$.

As it was shown in (23), the primitive idempotent $f$ in $\mathbb{C} \otimes C\ell_{1,3}$ generalized to the idempotent $f_B$ in $\mathbb{C} \otimes C\ell^B_{1,3}$ which we recall here in the notation from Section V:

$$\begin{aligned} f_B &= \tfrac{1}{4}(1+\gamma_0)\underset{B}{}(1+i\gamma_1\underset{B}{}\gamma_2)\\ &= \tfrac{1}{4}(1+\gamma_0)(1+i\gamma_1\gamma_2) + \tfrac{i}{4}(A_{12} + A_{12}\gamma_0 - A_{02}\gamma_1 + A_{01}\gamma_2). \end{aligned} \tag{A.5}$$

The fact that this idempotent is primitive follows from the algebra isomorphism $C\ell_{1,3} \cong C\ell^B_{1,3}$ [13, 24] extended to $\mathbb{C} \otimes C\ell_{1,3} \cong \mathbb{C} \otimes C\ell^B_{1,3}$. Notice that the generators $e_0, e_1, e_2, e_3$ satisfy the following relations in $C\ell^B_{1,3}$:

$$(e_i^2)_B = \begin{cases} +1 & \text{for } i = 0;\\ -1 & \text{for } i = 1,2,3, \end{cases} \quad \text{and} \quad e_i\underset{B}{}e_j + e_j\underset{B}{}e_i = 0 \quad \text{for} \quad i \neq j. \tag{A.6}$$

These relations generalize the defining relations satisfied by these generators in $C\ell_{1,3}$.

Thus, the four primitive idempotents (A.4) extend to four primitive mutually annihilating idempotents in $\mathbb{C} \otimes C\ell^B_{1,3}$, namely,

$$\begin{aligned} f_B^1 &= \tfrac{1}{4}(1+\gamma_0)\underset{B}{}(1+i\gamma_1\underset{B}{}\gamma_2)\\ &= \tfrac{1}{4}(1+\gamma_0)(1+i\gamma_1\gamma_2) + \tfrac{i}{4}(A_{12} + A_{12}\gamma_0 - A_{02}\gamma_1 + A_{01}\gamma_2),\\ f_B^2 &= \tfrac{1}{4}(1+\gamma_0)\underset{B}{}(1-i\gamma_1\underset{B}{}\gamma_2)\\ &= \tfrac{1}{4}(1+\gamma_0)(1-i\gamma_1\gamma_2) - \tfrac{i}{4}(A_{12} + A_{12}\gamma_0 - A_{02}\gamma_1 + A_{01}\gamma_2),\\ f_B^3 &= \tfrac{1}{4}(1-\gamma_0)\underset{B}{}(1+i\gamma_1\underset{B}{}\gamma_2)\\ &= \tfrac{1}{4}(1-\gamma_0)(1+i\gamma_1\gamma_2) - \tfrac{i}{4}(-A_{12} + A_{12}\gamma_0 - A_{02}\gamma_1 + A_{01}\gamma_2),\\ f_B^4 &= \tfrac{1}{4}(1-\gamma_0)\underset{B}{}(1-i\gamma_1\underset{B}{}\gamma_2)\\ &= \tfrac{1}{4}(1-\gamma_0)(1-i\gamma_1\gamma_2) + \tfrac{i}{4}(-A_{12} + A_{12}\gamma_0 - A_{02}\gamma_1 + A_{01}\gamma_2), \end{aligned} \tag{A.7}$$

where $f_B^1 = f_B$, $(f_B^i)^2 = f_B^i$, $f_B^i f_B^j = f_B^j f_B^i = 0$, for $i, j = 1, \ldots, 4, i \neq j$, and $f_B^1 + f_B^2 + f_B^3 + f_B^4 = 1$ in $\mathbb{C} \otimes C\ell^B_{1,3}$.

Thus, we can now consider the minimal ideal $S_B = (\mathbb{C} \otimes C\ell^B_{1,3})f_B$ in $\mathbb{C} \otimes C\ell^B_{1,3}$ (see (31)) whose elements are the $B$-spinors. It can be easily checked that a basis for this ideal generalizes the basis (A.1) for $S \subset \mathbb{C} \otimes C\ell_{1,3}$ and is given by

$$\begin{aligned} S_B = (\mathbb{C} \otimes C\ell^B_{1,3})f_B &= \mathrm{Span}_{\mathbb{C}}\{f_B,\, -e_1\underset{B}{}e_3\underset{B}{}f_B,\, e_3\underset{B}{}e_0\underset{B}{}f_B,\, e_1\underset{B}{}e_0\underset{B}{}f_B\}\\ &= \mathrm{Span}_{\mathbb{C}}\{f_B,\, -e_1\underset{B}{}e_3\underset{B}{}f_B,\, e_3\underset{B}{}f_B,\, e_1\underset{B}{}f_B\}. \end{aligned} \tag{A.8}$$

For completeness, we provide an explicit form for the symbolic (non-matrix) basis (A.8):

$f_B = \frac{1}{4}((1 + A_{12}i)1 + (1 + iA_{12})e_0 - iA_{02}e_1 + iA_{01}e_2 + ie_{12} + ie_{012}),$

$$
\begin{aligned}
-e_{1_B}e_{3_B}f_B &= \tfrac{1}{4}((iA_{23}-A_{13})1+(iA_{23}-A_{13})e_0+A_{03}e_1-iA_{03}e_2-(A_{01}-iA_{02})e_3\\
&\qquad -e_{13}+ie_{23}-e_{013}+ie_{023}),\\
e_{3_B}f_B &= \tfrac{1}{4}(-(A_{03}+iA_{03}A_{12}+iA_{01}A_{23}-iA_{13}A_{02})1+iA_{23}e_1-iA_{13}e_2+\\
&\qquad (1+iA_{12})e_3-iA_{23}e_{01}+iA_{13}e_{02}-(1+iA_{12})e_{03}-iA_{03}e_{12}+\\
&\qquad iA_{02}e_{13}-iA_{01}e_{23}-ie_{0123}+ie_{123}),\\
e_{1_B}f_B &= \tfrac{1}{4}(-(A_{01}-iA_{02})1+e_1-ie_2-e_{01}+ie_{02}),
\end{aligned}
\tag{A.9}
$$

where 1 denotes the unity of $\mathbb{C}\otimes C\ell_{1,3}^B$. Of course, when we set $A_{ij}=0$ for all the coefficients of the antisymmetric part $A$ appearing in (A.9), we obtain back the explicit basis for the ideal $S=(\mathbb{C}\otimes C\ell_{1,3})f$ shown in (A.1). Due to the relations (A.6), the gamma matrices (A.3) also represent the generators $e_0,e_1,e_2,e_3$ in the faithful and irreducible representation of the algebra $\mathbb{C}\otimes C\ell_{1,3}^B$ in the ideal $S_B$. This can be checked directly by computing these matrices in the explicit symbolic basis (A.9) with `CLIFFORD` [22].

## Appendix B. Additional terms in the quantum spinor fields

Recall from (31) that a $B$-spinor has the form

$$(\psi_B)_B(f_B)=(\psi)_B f+\psi(A)_B f+(\psi)_B f(A)+(\psi(A))_B f(A). \tag{B.1}$$

where the term $(\psi)_B f$ is the classical spinor field displayed in (15). The remaining terms in the above expression represent correction terms and are provided by:

(a) The term $\boxed{-4i(\psi(A))_B f(A)}$ is given by

$$
\begin{aligned}
&p\left[b^{013}\left(A_{01}\left(A_{01}A_{32}+A_{20}A_{31}+A_{12}A_{30}\right)+A_{12}A_{13}+A_{03}A_{20}\right)\right.\\
&\qquad +b^{023}\left(A_{02}\left(A_{01}A_{32}+A_{20}A_{31}+A_{12}A_{30}+A_{30}\right)+A_{12}A_{13}+A_{03}A_{20}\right)\\
&\qquad +b^{123}\left(A_{12}\left(A_{01}A_{32}+A_{20}A_{31}+A_{12}A_{30}\right)-A_{23}A_{20}-A_{31}A_{01}\right)\\
&\qquad \left.+b^{012}\left(A_{10}A_{01}+A_{20}A_{02}+A_{12}A_{12}\right)\right]\\
&+\gamma_0\left[p\left(A_{13}A_{01}-A_{23}A_{20}+2A_{12}A_{12}A_{13}+A_{23}A_{20}A_{12}+A_{23}A_{01}A_{12}\right)+sA_{12}\right]\\
&+\gamma_1\left[p\left(A_{12}A_{13}-A_{12}A_{23}-A_{03}A_{01}+A_{01}A_{20}A_{32}+A_{01}A_{20}A_{13}+A_{02}A_{12}A_{03}\right.\right.\\
&\qquad \left.\left.+A_{03}A_{12}A_{21}+A_{23}A_{01}A_{10}\right)+sA_{01}\right]\\
&+\gamma_2\left[p\left(A_{03}A_{20}+A_{01}A_{01}A_{32}+A_{13}A_{01}A_{02}+A_{13}A_{20}A_{02}\right)+sA_{02}\right]\\
&+\gamma_3\left[p\left(A_{01}A_{01}+A_{02}A_{02}+A_{02}A_{12}A_{13}+A_{12}A_{20}A_{10}+A_{02}A_{20}A_{12}+A_{01}A_{12}A_{13}\right)\right]\\
&+\gamma_{01}\left[p\left(b^{013}\left(A_{13}A_{20}+A_{21}A_{30}\right)+b^{023}\left(A_{03}A_{12}+A_{13}A_{20}\right)-b^{123}A_{23}A_{12}\right)\right]\\
&+\gamma_{02}\left[p\left(b^{013}A_{13}A_{01}+b^{023}A_{13}A_{01}+b^{123}A_{13}A_{21}\right)\right]\\
&+\gamma_{03}\left[p\left(b^{013}A_{01}A_{12}+b^{023}A_{02}A_{12}+b^{123}A_{12}A_{21}\right)\right]\\
&+\gamma_{12}\left[p\left(b^{013}A_{01}A_{30}+b^{023}A_{30}A_{01}+b^{123}\left(A_{13}A_{20}+A_{23}A_{01}\right)\right)\right]\\
&+\gamma_{31}\left[p\left(b^{013}A_{01}A_{20}+b^{023}A_{01}A_{20}+b^{123}A_{12}A_{20}\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
&+\gamma_{23}\left[p\left(b^{013}A_{01}A_{10}+b^{023}A_{01}A_{20}+b^{123}A_{12}A_{10}\right)\right]\\
&+\gamma_{023}A_{12}A_{01}\\
&+\gamma_{031}\left(A_{01}A_{12}+A_{12}A_{20}+A_{23}A_{01}\right)\\
&+\gamma_{012}\left(A_{03}A_{12}+A_{13}A_{20}\right).
\end{aligned}
\tag{B.2}
$$

(b) The term $\boxed{-4i(\psi(A))_{B}f}$ is given by

$$
\begin{aligned}
&\left[b^{023}\left(A_{03}+A_{02}\right)+b^{123}A_{23}+b^{012}A_{01}+b^{3}A_{01}A_{32}+b^{3}\left(A_{20}A_{31}+A_{03}A_{21}\right)\right.\\
&\qquad\left.+b^{0}A_{12}+b^{2}A_{01}+b^{1}A_{02}\right]\\
&+\gamma_{0}\left[b^{01}\left(\left(A_{01}+A_{20}\right)A_{12}+A_{20}\right)+b^{02}\left(A_{10}+A_{12}A_{20}\right)\right.\\
&\qquad+b^{03}\left(A_{01}A_{32}+A_{20}A_{32}+A_{12}A_{30}\right)+bA_{12}\\
&\qquad\left.+p\left(A_{12}+A_{20}A_{13}+A_{10}\left(A_{23}+A_{13}\right)+A_{30}A_{12}\right)\right]\\
&+\gamma_{1}\left[b^{01}\left(A_{20}\left(A_{01}+A_{02}\right)+A_{12}\left(A_{10}+1\right)\right)+bA_{20}\right.\\
&\qquad\left.+b^{12}\left(\left(A_{12}+A_{02}\right)A_{20}-A_{01}\right)A_{01}+b^{13}\left(A_{32}\left(A_{01}+A_{20}\right)+A_{12}A_{30}\right)\right]\\
&+\gamma_{2}\left[b^{02}A_{21}+b^{12}A_{20}+b^{23}\left(A_{01}A_{32}+A_{12}A_{31}+A_{20}A_{32}+bA_{01}\right.\right.\\
&\qquad\left.\left.+p\left(A_{12}A_{31}+2A_{02}\left(A_{01}A_{31}+A_{21}A_{30}\right)+A_{20}A_{20}\right)\right)\right]\\
&+\gamma_{3}\left[p\left(-A_{01}A_{01}-A_{02}A_{02}+A_{12}A_{12}\right)+b^{03}A_{21}+b^{13}\left(A_{01}A_{21}+A_{20}\right)\right.\\
&\qquad\left.+b^{23}A_{12}\left(A_{02}+A_{01}A_{13}\right)\right]\\
&+\gamma_{01}\left[b^{013}\left(A_{31}A_{20}+A_{12}A_{30}+A_{32}A_{01}\right)-b^{013}\left(A_{03}A_{12}+2A_{13}A_{20}\right)\right.\\
&\qquad\left.+b^{123}A_{23}\left(A_{12}+A_{02}\right)+b^{0}A_{20}+b^{1}A_{21}\right]\\
&+\gamma_{02}\left[b^{013}A_{31}A_{01}+b^{023}\left(A_{32}A_{01}+A_{20}A_{31}+A_{30}A_{12}\right)+b^{123}A_{32}A_{01}\right.\\
&\qquad\left.+b^{012}A_{21}A_{01}+(b^{0}+b^{2})A_{01}\right]\\
&+\gamma_{03}\left(2b^{012}A_{20}+b^{3}A_{12}\right)\\
&+\gamma_{12}\left[b^{1}A_{01}+b^{2}A_{02}+b^{012}A_{12}+b^{023}\left(A_{13}A_{01}+A_{03}A_{20}\right)\right.\\
&\qquad\left.+b^{123}\left(A_{30}A_{01}+A_{30}A_{12}+A_{20}A_{31}\right)\right]\\
&-\gamma_{31}\left(b^{013}A_{01}A_{20}+b^{012}A_{01}A_{20}+b^{123}A_{01}\right)\\
&+\gamma_{23}\left[b^{023}A_{12}+b^{123}\left(A_{10}A_{12}+A_{20}+A_{02}A_{01}\right)+b^{3}A_{10}\right]\\
&+\gamma_{023}\left[b^{03}A_{10}+b^{23}A_{12}+p\left(A_{20}A_{12}+A_{02}+A_{21}A_{10}\right)\right]\\
&+\gamma_{013}\left[p\left(2A_{20}A_{02}+A_{01}\right)+b^{03}A_{02}+b^{13}A_{12}\right]\\
&+\gamma_{012}\left[b^{01}A_{01}+b^{02}A_{02}+b^{12}A_{12}+p\left(A_{02}A_{23}+A_{13}A_{20}+A_{21}A_{20}\right)\right]\\
&+\gamma_{123}\left[p\left(A_{21}+A_{20}A_{20}\right)+b^{23}A_{20}+b^{13}A_{10}\right]\\
&+\gamma_{0123}\left(b^{013}A_{10}+b^{023}A_{20}+b^{123}A_{12}\right)
\end{aligned}
\tag{B.3}
$$

(c) The term $\boxed{-4i\psi_{B}f}$ is given by

$$
\begin{aligned}
&\left[b^{012}\left(A_{21}A_{12}-1\right)+b^{013}A_{31}A_{02}+b^{023}A_{31}+b^{123}\left(A_{30}-A_{32}A_{20}\right)\right.\\
&\qquad\left.+p\left(A_{02}\left(A_{23}+A_{13}+A_{01}A_{12}\right)+A_{01}\left(A_{32}+A_{31}\right)+\left(A_{21}-1\right)A_{03}\right)\right]\\
&+\gamma_{0}\left[b^{01}\left(A_{21}A_{01}+A_{02}+A_{10}\right)+b^{02}\left(A_{21}A_{20}+A_{01}+A_{20}\right)\right.
\end{aligned}
$$

$$
\begin{aligned}
&+b^{03}\left(A_{02}A_{31}+A_{23}A_{01}+A_{30}\right)+b^{31}\left(A_{21}A_{31}+A_{23}\right)\\
&+p\left(A_{30}A_{20}+A_{32}A_{01}+A_{21}A_{20}A_{13}+A_{30}A_{21}A_{12}+A_{30}A_{20}\right.\\
&\qquad\left.+A_{32}A_{01}+A_{21}A_{12}\left(A_{02}+A_{30}\right)+b^{023}A_{23}A_{21}\right)\\
&+b^{12}\left(A_{21}A_{12}-1\right)+b^{23}\left(A_{31}-A_{32}A_{21}\right)]\\
+\gamma_1&\left[b^{01}\left(A_{20}A_{10}+A_{12}+2\right)+b^{02}\left(A_{20}A_{02}-1\right)+b^{03}\left(A_{32}-A_{30}A_{02}\right)\right.\\
&\qquad+b^{31}\left(A_{01}A_{32}+A_{12}A_{30}+A_{30}\right)\\
&+b^{12}\left(A_{20}A_{12}+A_{01}+A_{02}\right)+b^{23}\left(A_{30}+A_{32}A_{20}\right)\\
&\left.+p\left(A_{32}A_{01}A_{02}+A_{30}A_{20}A_{12}\right)\right]\\
+\gamma_2&\left[b^{01}\left(A_{10}A_{01}-1\right)+b^{02}\left(A_{20}A_{01}+A_{12}+2\right)+b^{03}\left(A_{30}A_{01}+A_{13}\right)\right.\\
&+b^{31}\left(A_{10}A_{31}+A_{30}\right)+b^{12}\left(A_{10}A_{12}+A_{02}\right)+b^{23}\left(A_{30}A_{21}+A_{31}A_{20}\right)\\
&\left.+p\left(A_{01}\left(A_{12}A_{30}+A_{32}A_{10}\right)+A_{23}\right)\right]\\
+\gamma_3&\left(b^{123}A_{23}+2b^{03}+2b^{13}A_{10}+2b^{23}A_{20}+pA_{20}\right)\\
+\gamma_{01}&\left[b^{013}A_{03}A_{12}+b^{012}A_{03}A_{12}+b^{123}\left(A_{13}+A_{23}A_{12}\right)+b^{023}A_{30}\right.\\
&\left.+p\left(A_{20}\left(A_{32}A_{10}+A_{31}A_{20}\right)+A_{31}+A_{12}A_{23}\right)\right]\\
+\gamma_{02}&\left[b^{013}A_{03}+b^{012}A_{10}+b^{123}A_{32}+b^{023}\left(A_{30}A_{21}+A_{23}A_{01}+A_{13}A_{20}\right)\right.\\
&\left.+p\left(A_{10}\left(A_{32}A_{10}+A_{31}A_{20}\right)+A_{23}\right)\right]\\
+\gamma_{03}&\left(b^{123}A_{32}\right)\\
+\gamma_{12}&[b^{012}(A_{21}+A_{02}A_{01})+b^{023}A_{32}+b^{123}A_{30}(A_{21}+A_{10})\\
&+b^{013}(A_{31}+A_{03}A_{10})+p(A_{30}A_{12}+A_{23}A_{10}+A_{13}A_{20})]\\
+\gamma_{31}&\left(b^{013}A_{12}+pA_{02}A_{21}\right)\\
+\gamma_{012}&\left[b^{01}A_{01}+b^{02}A_{02}+b^{12}A_{12}+p\left(A_{02}A_{23}+A_{13}A_{20}+A_{21}A_{20}\right)\right]\\
+\gamma_{013}&\left[p\left(A_{20}A_{21}+A_{10}\right)+b^{03}A_{20}-b^{23}+b^{31}A_{12}\right]\\
+\gamma_{023}&\left[b^{03}A_{10}+b^{31}+b^{23}A_{12}+p\left(A_{10}A_{12}+A_{02}+A_{21}A_{01}\right)\right]\\
+\gamma_{123}&\left[p(A_{21}\left(A_{30}+A_{32}\right)+A_{10}A_{32})+b^{23}A_{02}+b^{31}A_{10}\right]\\
+\gamma_{0123}&\left(b^{023}A_{02}+b^{123}A_{12}\right)
\end{aligned}
\tag{B.4}
$$

(d) The term $\boxed{-4i\psi_{_B}f(A)}$ is given by

$$
\begin{aligned}
&\left[b^0A_{12}+b^2A_{10}+b^1A_{02}\right]\\
+\gamma_0&\left[b^{01}\left(A_{12}A_{01}+A_{02}A_{21}+A_{20}\right)+b^{02}\left(A_{10}+A_{12}A_{20}\right)\right.\\
&+b^{03}\left(A_{01}A_{32}+A_{23}A_{02}+A_{12}A_{30}\right)\\
&\left.+bA_{12}+pA_{12}\left(A_{20}A_{13}+A_{32}A_{01}+A_{30}A_{12}+A_{10}A_{13}\right)\right]\\
+\gamma_1&\left[b^{01}\left(A_{20}\left(A_{01}+A_{02}\right)+A_{12}\left(A_{01}+1\right)\right)+b^{12}\left(A_{10}+A_{02}A_{20}+A_{12}A_{20}\right)\right.\\
&+b^{13}\left(A_{01}A_{32}+A_{23}A_{02}+A_{12}A_{30}\right)+bA_{20}\\
&\left.+p\left(A_{12}A_{23}+A_{10}A_{30}+A_{02}A_{01}A_{23}\right)\right]\\
+\gamma_2&\left[b^{02}A_{21}+b^{12}A_{20}+b^{23}\left(A_{12}A_{30}+A_{32}A_{01}+A_{20}A_{32}\right)\right.\\
&\left.+p\left(2A_{10}A_{31}+2A_{12}A_{30}+A_{20}\right)+A_{12}A_{31}\right]
\end{aligned}
$$

$$
\begin{aligned}
&+\gamma_3 \left[A_{20}A_{02} + A_{10}A_{01} + A_{12}A_{12} + b^{03}A_{21} + b^{13}(A_{20} + A_{01}A_{21}) + b^{23}A_{01}\right] \\
&+\gamma_{01} \left[b^{013}\left(A_{03}A_{12} + A_{31}A_{20}\right) + b^{123}A_{23}(A_{12} + A_{02}\right) + b^{012}\left(A_{10} + A_{20}A_{21}\right) \\
&\qquad + b^{013}A_{32}A_{01} + b^{0}A_{20} + b^{1}A_{21}\big] \\
&+\gamma_{02} \left[b^{013}A_{01}A_{31} + b^{123}A_{32}A_{01} + b^{012}\left(A_{20} + A_{21}A_{01}\right)\right. \\
&\qquad \left. + b^{023}\left(A_{23}A_{01} + A_{20}A_{31} + A_{30}A_{12}\right) + b^{0}A_{01} + b^{2}A_{01}\right] \\
&+\gamma_{03}\left(b^{012}A_{20}\right) \\
&+\gamma_{12} \left[b^{013}A_{01}A_{31} + b^{123}\left(A_{20}A_{31} + A_{30}A_{12} + A_{32}A_{01}\right)\right. \\
&\qquad \left. + b^{023}\left(A_{13}A_{01} + A_{03}A_{20}\right) + b^{1}A_{01} + b^{2}A_{02}\right] \\
&+\gamma_{13}\left(b^{013}A_{12} + b^{123}A_{01} + b^{013}A_{02} + b^{3}A_{02}\right) \\
&+\gamma_{23}\left[b^{123}\left(A_{01}(A_{02} + A_{21}) + A_{20}\right) + b^{023}A_{12}\right]
\end{aligned}
\tag{B.5}
$$

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