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RINGS OF INVARIANTS FOR SALINGAROS' VEE GROUPS

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Rings of Invariants for Salingaros' Vee Groups*

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Abstract: Generators of rings of invariants and the ideal of relations are computed for Salingaros's vee groups of orders 4, 8, and 16.

1 Introduction

This article begins with a summary of major theorems from [3] which provide a method of finding an ideal of relations for representations of finite groups. In particular, these methods are applied to Salingaros' vee groups of orders 4, 8, and 16 whose irreducible representations were computed in [2]. Salingaros' vee groups are discussed in [5, 6, 7]. The main purpose of this article is to compute generators for the rings of invariants for the Salingaros' vee groups, and then compute the ideal of relations for these groups. In this article all computations are performed with a package [1] for Maple.

2 Summary of major theorems

In this section, we discuss major theorems needed to compute the ring of invariants $k[x_1, \ldots, x_n]^G$ and the ideal of relations for a finite group G. These results will be used in later sections where for G we will take the Salingaros' vee groups of orders 4, 8, and 16.

First, we introduce the definition of *Reynolds' operator* as follows.

Definition 1. Given a finite matrix group $G \subset \mathrm{GL}(n, k)$, the Reynolds' operator of G is the map $R_G : k[x_1, \ldots, x_n] \rightarrow k[x_1, \ldots, x_n]$ defined by the formula

$$R_G(f)(\mathbf{x}) = \frac{1}{|G|} \sum_{g \in G} f(A \cdot \mathbf{x}) \tag{1}$$

for $f(\mathbf{x}) \in k[x_1, \ldots, x_n]$. Here, $\mathbf{x}$ denotes a column vector $[x_1, \ldots, x_n]^t$ and $A \cdot \mathbf{x}$ denotes the action of a matrix A on $\mathbf{x}$ by the left matrix multiplication (see [3, P. 329] for more details).

The following theorem can be used to find finitely many polynomial invariants that generate the ring of invariants $k[x_1, \ldots, x_n]^G$ of the group G. Notation is taken from [3] where all proofs can be found.

Theorem 1. *Given a finite matrix group $G \subset \mathrm{GL}(n, k)$, we have*

$$k[x_1, \ldots, x_n]^G = k[R_G(x^\beta) : |\beta| \leq |G|]. \tag{2}$$

In particular, $k[x_1, \ldots, x_n]^G$ is generated by finitely many homogeneous invariants.

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Theorem 1 guarantees that there are finitely many invariants $f_1, \dots, f_m$ such that $k[x_1, \dots, x_n]^G = k[f_1, \dots, f_m]$. Suppose that g_1 and g_2 are polynomials in $k[y_1, \dots, y_m]$, then

$$g_1(f_1, \dots, f_m) = g_2(f_1, \dots, f_m) \Longleftrightarrow h(f_1, \dots, f_m) = 0, \tag{3}$$

where $h = g_1 - g_2$. It follows that uniqueness of the algebraic relations fails if and only if a nonzero polynomial $h \in k[y_1, \dots, y_m]$ exists such that

$$h(f_1, \dots, f_m) = 0. \tag{4}$$

Such nonzero polynomial h is a *nontrivial algebraic relation* among $f_1, \dots, f_m$. The following important definition introduces an *ideal of relations* also called a *syzygy ideal*.

Definition 2. If we let $F = (f_1, \dots, f_m)$, then the set

$$I_F = \{h \in k[y_1, \dots, y_m] : h(f_1, \dots, f_m) = 0 \quad \text{in} \quad k[x_1, \dots, x_n]\} \tag{5}$$

records all algebraic relations among $f_1, \dots, f_m$.

The set I_F has the following properties.

Proposition 1. *If $k[x_1, \dots, x_n]^G = k[f_1, \dots, f_m]$, let $I_F \subset k[y_1, \dots, y_m]$ be as in* (5)*. Then:*

(i) I_F is a prime ideal of $k[y_1, \dots, y_m]$.

(ii) Suppose that $f \in k[x_1, \dots, x_n]^G$ and that $f = g(f_1, \dots, f_m)$ is one representation of f in terms of $f_1, \dots, f_m$. Then all such representations are given by

$$f = g(f_1, \dots, f_m) + h(f_1, \dots, f_m), \tag{6}$$

as h varies over I_F.

Proposition 2. *If $k[x_1, \dots, x_n]^G = k[f_1, \dots, f_m]$, let $I_F \subset k[y_1, \dots, y_m]$ be the ideal of relations. Then there is a ring isomorphism*

$$k[y_1, \dots, y_m]/I_F \cong k[x_1, \dots, x_n]^G \tag{7}$$

between the quotient ring of I_F and the ring of invariants.

Proposition 3. *If $k[x_1, \dots, x_n]^G = k[f_1, \dots, f_m]$, consider the ideal*

$$J_F = \langle f_1 - y_1, \dots, f_m - y_m \rangle \subset k[x_1, \dots, x_n, y_1, \dots, y_m]. \tag{8}$$

(i) I_F is the n-th elimination ideal of J_F. Thus, $I_F = J_F \cap k[y_1, \dots, y_m]$.

(ii) Fix a monomial order in $k[x_1, \dots, x_n, y_1, \dots, y_m]$ where any monomial involving one of $x_1, \dots, x_n$ is greater than all monomials in $k[y_1, \dots, y_m]$ and let G be a Groebner basis of J_F. Then $G \cap k[y_1, \dots, y_m]$ is a Groebner basis for I_F in the monomial order induced on $k[y_1, \dots, y_m]$.

Example 1. The invariants of $C_2 = \{\pm I_2\} \subset \mathrm{GL}(2, k)$ are given by

$$k[x_1, x_2]^{C_2} = k[x_1^2, x_2^2, x_1 x_2]. \tag{9}$$

Let $F = (x_1^2, x_2^2, x_1x_2)$ and let the new variables be u, v, w. Then the ideal of relations is obtained by eliminating x_1, x_2 from the equations

$$\begin{aligned} u &= x_1^2, \\ v &= x_2^2, \\ w &= x_1x_2. \end{aligned} \tag{10}$$

If we use the lex order with $x_1 > x_2 > u > v > w$, then a Groebner basis for the ideal $J_F = \langle u - x_1^2, v - x_2^2, w - x_1x_2 \rangle$ consists of the polynomials

$$x_1^2 - u,\; x_1x_2 - w,\; x_1v - x_2w,\; x_1w - x_2v,\; x_2^2 - v,\; uv - w^2. \tag{11}$$

It follows from Proposition 3 that $I_F = \langle uv - w^2 \rangle$. This says that all relations between x_1^2, x_2^2, and x_1x_2 are generated by the obvious relation $x_1^2x_2^2 = (x_1x_2)^2$. Then, Proposition 2 shows that the ring of invariants can be written as

$$k[x_1, x_2]^{C_2} \cong k[u, v, w]/\langle uv - w^2 \rangle. \tag{12}$$

Example 2. The cyclic matrix group $C_4 \subset \mathrm{GL}(2, k)$ of order 4 can be generated by

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{13}$$

and

$$k[x_1, x_2]^{C_4} = k[x_1^2 + x_2^2,\; x_1^3x_2 - x_1x_2^3,\; x_1^2x_2^2]. \tag{14}$$

Putting $F = (x_1^2 + x_2^2, x_1^3x_2 - x_1x_2^3, x_1^2x_2^2)$, the ideal $I_F \subset k[u, v, w]$ is given by $I_F = \langle u^2w - v^2 - 4w^2 \rangle$. By Proposition 2, the ring of invariants can be written as

$$k[x_1, x_2]^{C_4} \cong k[u, v, w]/\langle u^2w - v^2 - 4w^2 \rangle. \tag{15}$$

3 Rings of invariants for the Salingaros' groups

In this section we describe the process of computing the syzygy ideals of Salingaros' vee groups $G_{p,q} \subset C\ell_{p,q}$ of orders 4, 8, and 16 using the above Propositions and `SymGroupAlgebra`, a package for Maple [1].

3.1 The groups of order 4

The vee groups $G_{1,0}$ and $G_{0,1}$ of order 4 are isomorphic to $D_2 \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \cong C_2 \times C_2$ and $\mathbb{Z}_4 \cong C_4$, respectively. Computation of their irreducible representations of degree 2 and higher can be found in [2]. Now consider results for the above groups separately.

First, consider the group $G_{1,0} \cong C_2 \times C_2$ which we can write as the matrix group $G_{1,0} = \{E, A, B, C\}$ where

$$E = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

$$B = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{16}$$

The ring of invariants $k[x_1, x_2, x_3, x_4]^{G_{1,0}}$ is computed using Theorem 1. It is generated by six polynomials. That is,

$$k[x_1, x_2, x_3, x_4]^{G_{1,0}} = k[x_1^2,\ x_2^2,\ x_3^2,\ x_4^2,\ x_1x_2,\ x_3x_4]. \tag{17}$$

Then the two non-trivial relations on these polynomials are exactly the two polynomials in $y_1, \ldots, y_6$ generating I_F as shown below.

$$I_F = \langle y_2^2 - y_1y_3,\ y_5^2 - y_4y_6 \rangle. \tag{18}$$

Now consider the group $G_{0,1} \cong C_4$ which we can realize as the matrix group $G_{0,1} = \{E, A, B, C\}$ where

$$E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{19}$$

The ring of invariants $k[x_1, x_2]^{G_{0,1}}$ is again computed using Theorem 1. It is generated by three polynomials. That is,

$$k[x_1, x_2, x_3, x_4]^{G_{1,0}} = k[x_1^2x_2^2,\ x_2^2 + x_1^2,\ x_1^3x_2 - x_1x_2^3]. \tag{20}$$

Then there is only one non-trivial polynomial relation in $y_1, \ldots, y_3$ generating I_F as shown below.

$$I_F = \langle y_2y_3^2 - y_1^2 - 4y_2^2 \rangle. \tag{21}$$

3.2 The groups of order 8

Since the vee groups $G_{2,0}$, $G_{1,1}$ and $G_{0,2}$ of order 8 are isomorphic to D_4, D_4 and Q_4, respectively, their character tables and their representations are computed in [2]. Now we present results for each of these groups separately.

First, let's consider $G_{2,0}$. Using a degree 2 representation from [2], the ring of invariants $k[x_1, x_2]^{G_{2,0}}$ is found to be generated by twenty four polynomials while the ideal of relations I_F happens to be the zero ideal $I_F = \langle 0 \rangle$. This means that there are no non-trivial relations among the invariants. The quotient ring $k[y_1, \ldots, y_{24}]/I_F$ is essentially isomorphic to $k[y_1, \ldots, y_{24}]$. Intuitively, the quotient ring $k[y_1, \ldots, y_{24}]/I_F$ is a simplified version of $k[y_1, \ldots, y_{24}]$ where the elements of I_F are ignored.

Now consider $G_{0,2}$. The ring of invariants $k[x_1, x_2]^{G_{0,2}}$ is generated by five polynomials. That is,

$$k[x_1, x_2]^{G_{0,2}} = k[x_1^2x_2^2,\ x_2^4 + x_1^4,\ x_1x_2^5 - x_1^5x_2,\ x_1^2x_2^6 + x_1^6x_2^2,\ x_1^5x_2 - x_1x_2^5]. \tag{22}$$

Then, the three non-trivial relations on these polynomials are exactly the two polynomials in $y_1, \ldots, y_5$ generating I_F as shown below.

$$I_F = \langle y_3^2 - y_1y_5 + 2y_2y_5,\ y_4^2 - y_1 - 2y_2,\ y_5^2 - y_2 \rangle. \tag{23}$$

3.3 The groups of order 16

The vee groups $G_{3,0}$, $G_{2,1}$, $G_{1,2}$ and $G_{0,3}$ of order 16 are denoted by Salingaros as S_1, Ω_1, S_1, and Ω_2, respectively. Their character tables and their irreducible representations are computed in [2]. Now consider results for each of these groups separately.

First consider $G_{3,0}$ or $G_{1,2}$. Then, using the degree 2 representations from [2], rings of invariants $k[x_1,x_2]^{G_{3,0}}$ and $k[x_1,x_2]^{G_{1,2}}$ are computed. Each of them is generated by thirteen polynomials. Later, we compute the corresponding syzygy ideals. Then, the sixty one non-trivial relations on these polynomials are exactly the sixty one polynomials in $y_1,\ldots,y_{13}$ generating I_F as shown below.

$$\begin{aligned}I_F = \langle & y_1y_{12}y_{13}-y_2y_{10}-y_2y_{11}+y_4y_{11},\ y_2^2-y_1y_3+y_3y_5-2y_5^2,\ y_2y_3-y_1y_4+y_2y_5-y_4y_5,\ y_3^2-y_1y_5-2y_5^2,\\ & y_2y_4-y_1y_5-y_3y_5,\ y_3y_4-y_2y_5-y_4y_5,\ y_4^2-y_3y_5-2y_5,\ y_2y_6-y_1y_7+y_3y_9-2y_5y_9,\ y_3y_6-y_1y_8+y_2y_9-\\ & y_4y_9,\ y_4y_6-y_1y_9-y_3y_9,\ y_5y_6-y_2y_9,\ y_6^2-y_1y_{10}+y_3y_{11}-2y_5y_{11},\ y_2y_7-y_1y_8+y_2y_9-y_4y_9,\ y_3y_7-y_1y_9\\ & -2y_5y_9,\ y_4y_7-y_2y_9-y_4y_9,\ y_5y_7-y_3y_9,\ y_6y_7-y_2y_{10},\ y_7^2-y_1y_{11}-2y_5y_{11},\ y_2y_8-y_1y_9-y_3y_9,\ y_3y_8-y_2y_9\\ & -y_4y_9,\ y_4y_8-y_3y_9-2y_5y_9,\ y_5y_8-y_4y_9,\ y_6y_8-y_1y_{11}-y_3y_{11},\ y_7y_8-y_2y_{11}-y_4y_{11},\ y_8^2-y_3y_{11}-2y_5y_{11},\\ & y_6y_9-y_2y_{11},\ y_7y_9-y_3y_{11},\ y_8y_9-y_4y_{11},\ y_9^2-y_5y_{11},\ y_3y_{10}-y_1y_{11}-2y_5y_{11},\ y_4y_{10}-y_2y_{11}-y_4y_{11},\ y_5y_{10}\\ & -y_3y_{11},\ y_6y_{10}-y_1y_{12}+y_2y_{13}-y_4y_{13},\ y_7y_{10}-y_1y_{13}-2y_5y_{13},\ y_8y_{10}-y_2y_{13}-y_4y_{13},\ y_9y_{10}-y_3y_{13},\ y_{10}^2-\\ & y_1-2y_5,\ y_6y_{11}-y_2y_{13},\ y_7y_{11}-y_3y_{13},\ y_8y_{11}-y_4y_{13},\ y_9y_{11}-y_5y_{13},\ y_{10}y_{11}-y_3,\ y_{11}^2-y_5,\ y_2y_{12}-y_1y_{13}-\\ & y_3y_{13},\ y_3y_{12}-y_2y_{13}-y_4y_{13},\ y_4y_{12}-y_3y_{13}-2y_5y_{13},\ y_5y_{12}-y_4y_{13},\ y_6y_{12}-y_1-y_3,\ y_7y_{12}-y_2-y_4,\ y_8y_{12}\\ & -y_3-2y_5,\ y_9y_{12}-y_4,\ y_{10}y_{12}-y_6-y_8,\ y_{11}y_{12}-y_8,\ y_{12}^2-y_{10}-2y_{11},\ y_6y_{13}-y_2,\ y_7y_{13}-y_3,\ y_8y_{13}-y_4,\\ & y_9y_{13}-y_5,\ y_{10}y_{13}-y_7,\ y_{11}y_{13}-y_9,\ y_{13}-y_{11}\rangle \end{aligned} \tag{24}$$

Now consider $G_{2,1}$. In a similar manner, the ring of invariants $k[x_1,x_2]^{G_{2,1}}$ is computed using Theorem 1. It has been found to be generated by twenty four polynomials. The ideal of relations I_F happens to be the zero ideal.

For the group $G_{0,3}$, the ring of invariants $k[x_1,x_2]^{G_{0,3}}$ is generated by twenty polynomials. Then the one hundred and five non-trivial relations on these polynomials are exactly the one hundred and five polynomials in $y_1,\ldots,y_{20}$ generating I_F as shown below.

$$
\begin{gathered}
I_F = \langle y_{11}^4 - y_1 y_5^2 y_{17} + 4y_3 y_5^2 y_{17} - 6y_5^3 y_{17},\ 4y_5^2 y_{18} y_{20} + y_{11}^3 - y_4 y_5 y_{15},\ y_1 y_{11}^2 - 2y_5 y_{11}^2 - y_1 y_4 y_{13} + 4y_2 y_5 y_{13} - \\
2y_4y_5y_{13},\ y_2y_{11}^2 - y_1y_5y_{13} + 3y_3y_5y_{13} - 4y_5^2y_{13},\ y_3y_{11}^2 + 2y_5y_{11}^2 - y_2y_5y_{13} + y_4y_5y_{13},\ y_4y_{11}^2 - y_3y_5y_{13} + \\
2y_5^2y_{13},\ y_{11}^2y_{20} - y_3y_5 + 2y_5^2,\ y_1y_{16}y_{20} - y_{11}^2 - y_2y_{12},\ y_2y_{16}y_{20} - y_1y_{13} + y_3y_{13} - 2y_5y_{13},\ 2y_1y_{17}y_{20} + y_{11}^2 - \\
2y_2y_{13} + y_4y_{13},\ y_2y_{17}y_{20} - y_3y_{13} + y_5y_{13},\ 2y_3y_{17}y_{20} - y_{11}^2 - y_4y_{13},\ y_4y_{17}y_{20} - y_5y_{13},\ 4y_5y_{17}y_{20} + y_{11}^2 - y_4y_{13}, \\
y_1y_{18}y_{20} - 2y_5y_{18}y_{20} - y_2y_{15} + y_4y_{15},\ y_2y_{18}y_{20} - y_3y_{15} + y_5y_{15},\ y_3y_{18}y_{20} + 2y_5y_{18}y_{20} - y_4y_{15},\ y_4y_{18}y_{20} - \\
y_5y_{15},\ y_1y_{19}y_{20} - y_2y_{16} - y_2y_{17} + y_4y_{17},\ y_7y_{19}y_{20} - y_2y_{18} - 2y_4y_{18},\ y_2^2 - y_1y_3 + y_3y_5 - 2y_5^2,\ y_2y_3 - y_1y_4 + \\
y_2y_5 - y_4y_5,\ y_3^2 - y_1y_5 - 2y_5^2,\ y_2y_4 - y_1y_5 - y_3y_5,\ y_3y_4 - y_2y_5 - y_4y_5,\ y_4^2 - y_3y_5 - 2y_5^2,\ y_2y_7 - y_1y_9 - y_4y_{11}, \\
y_3y_7 - y_1y_{11} - y_3y_{11} - 2y_5y_{11},\ y_4y_7 - y_2y_{11} - 2y_4y_{11},\ y_5y_7 - y_3y_{11} - y_5y_{11},\ y_7^2 - y_{11}^2 - y_2y_{12} + y_4y_{13},\ y_2y_9 \\
- y_1y_{11} - y_3y_{11},\ y_3y_9 - y_2y_{11} - y_4y_{11},\ y_4y_9 - y_3y_{11} - 2y_5y_{11},\ y_5y_9 - y_4y_{11},\ y_7y_9 - y_1y_{13} + y_3y_{13},\ y_9^2 - y_2y_{13} \\
+ y_4y_{13},\ y_7y_{11} + y_{11}^2 - y_2y_{13} + y_4y_{13},\ y_9y_{11} - y_3y_{13} + 2y_5y_{13},\ y_3y_{12} - y_1y_{13} + y_3y_{13} - 2y_5y_{13},\ y_4y_{12} - y_2y_{13}, \\
y_5y_{12} - y_3y_{13} + y_5y_{13},\ y_7y_{12} - y_1y_{15} - y_5y_{15},\ y_9y_{12} - y_2y_{15},\ y_{11}y_{12} - y_3y_{15} + y_5y_{15},\ y_{12}^2 - y_1y_{16} + y_3y_{17} - \\
2y_5y_{17},\ y_7y_{13} - y_3y_{15} - y_5y_{15},\ y_9y_{13} - y_4y_{15},\ y_{11}y_{13} - y_5y_{15},\ y_{12}y_{13} - y_1y_{17} - y_3y_{17},\ y_{13}^2 - y_3y_{17} - 2y_5y_{17}, \\
y_7y_{15} - y_2y_{16} + 2y_4y_{17},\ y_9y_{15} - y_1y_{17} + 2y_5y_{17},\ y_{11}y_{15} - y_2y_{17} + y_4y_{17},\ y_{12}y_{15} - y_1y_{18} - y_3y_{18},\ y_{13}y_{15} - \\
y_3y_{18} - 2y_5y_{18},\ y_{15}^2 - y_1y_{20} + 2y_5y_{20},\ y_3y_{16} - y_1y_{17} - 2y_5y_{17},\ y_4y_{16} - y_2y_{17} - y_4y_{17},\ y_5y_{16} - y_3y_{17},\ y_7y_{16} - \\
y_1y_{18} - y_3y_{18} - 2y_5y_{18},\ y_9y_{16} - y_2y_{18} - y_4y_{18},\ y_{11}y_{16} - y_3y_{18},\ y_{12}y_{16} - y_1y_{19} + y_2y_{20} - y_4y_{20},\ y_{13}y_{16} - y_2y_{20} \\
- y_4y_{20},\ y_{15}y_{16} - y_7y_{19} + y_9y_{20},\ y_{16}^2 - y_1 - 2y_5,\ y_7y_{17} - y_3y_{18} - y_5y_{18},\ y_9y_{17} - y_4y_{18},\ y_{11}y_{17} - y_5y_{18},\ y_{12}y_{17} \\
- y_2y_{20},\ y_{13}y_{17} - y_4y_{20},\ y_{15}y_{17} - y_9y_{20},\ y_{16}y_{17} - y_3,\ y_{17}^2 - y_5,\ y_7y_{18} - y_1y_{20} + y_3y_{20},\ y_9y_{18} - y_2y_{20} + y_4y_{20}, \\
y_{11}y_{18} - y_3y_{20} + 2y_5y_{20},\ y_{12}y_{18} - y_7y_{19} + 2y_9y_{20},\ y_{13}y_{18} - y_9y_{20},\ y_{15}y_{18} - y_2 + y_4,\ y_{16}y_{18} - y_7 + y_{11},\ y_{17}y_{18} \\
- y_{11},\ y_{18}^2 - y_{16}y_{20} + 2y_{17}y_{20},\ y_2y_{19} - y_1y_{20} - y_3y_{20},\ y_3y_{19} - y_2y_{20} - y_4y_{20},\ y_4y_{19} - y_3y_{20} - 2y_5y_{20},\ y_5y_{19} - \\
y_4y_{20},\ y_9y_{19} - y_7y_{20} - y_{11}y_{20},\ y_{11}y_{19} - y_9y_{20},\ y_{12}y_{19} - y_1 - y_3,\ y_{13}y_{19} - y_3 - 2y_5,\ y_{15}y_{19} - y_7 - y_{11},\ y_{16}y_{19} \\
- y_{12} - y_{13},\ y_{17}y_{19} - y_{13},\ y_{18}y_{19} - y_{15},\ y_{19}^2 - y_{16} - 2y_{17},\ y_{12}y_{20} - y_2,\ y_{13}y_{20} - y_4,\ y_{15}y_{20} - y_9,\ y_{20}^2 - y_{17},\ y_6 \\
+ y_7,\ y_8 + y_9,\ y_{10} + y_{11},\ y_{14} + y_{15}\rangle.
\end{gathered} \tag{25}
$$

4 Conclusions

In this article, the theory of invariants of finite groups has been applied to irreducible representations of Salingaros' vee groups. In particular, we have discussed the rings of invariants and the syzygy ideals for Salingaros' vee groups of orders 4, 8 and 16.

In Section 2, the author summarized important results regarding the theory of invariants of finite group which she used in Section 3 to compute the rings of invariants and the syzygy ideal for the Salingaros' vee groups of orders 4, 8 and 16.

All computations were performed with a help of the Maple package `SymGroupAlgebra`.

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