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STEINHAUS PROPERTY IN BANACH LATTICES

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Abstract. We prove that a strictly monotone order continuous Banach lattice X over a non-atomic measure space has the Steinhaus property, that is for every quasi-finite set $A \subset X$ there is a dense set $Y \subset X$ such that for every $y \in Y$ and for every positive integer n there exists an open ball centered at y which contains exactly n points of A. We present several examples of spaces satisfying and failing the Steinhaus property. We also deliver detailed proofs of some important and known results connected with the Steinhaus property.

1. Introduction

Hugo Steinhaus in 1957 in a polish journal for high school teachers "Matematyka" raised a question [3] of existence of a circle in the plane surrounding exactly n points of the integer lattice $\mathbb{Z} \times \mathbb{Z}$, $n \in \mathbb{N}$. The answer to that question is affirmative and can be found in [4] or [1]. In fact, it is possible to show that there exists a dense set $Y \subset \mathbb{R}^2$ such that for every $y \in Y$ and for every integer n there is a circle surrounding exactly n points of $\mathbb{Z} \times \mathbb{Z}$. That problem have been generalized by P. Zwoleński in 2011 [5] in the following way. An infinite set A in a metric space X is said to be *quasi finite* if every open ball in X contains finitely many elements of A. We say that a Banach space $(X, \|\cdot\|_X)$ has the Steinhaus property if for every quasi-finite set $A \subset X$ there exists a dense set $Y \subset X$ such that for every $y \in Y$ and every $n \in \mathbb{N}$ there exists an open ball B centered at y which contains exactly n elements of A. Zwoleński showed that every Hilbert space has the Steinhaus property.

In a preprint from 2013, T. Kochanek found an equivalent condition for the Steinhaus property in Banach spaces. He also showed that the L_p spaces over a non-atomic measure space have the Steinhaus property if $1 \leqslant p < \infty$ [2]. We present a generalization of that result in section 4. In section 3 give detailed proofs of some Kochanek's results.

2. Preliminaries

Let X be a Banach space over $\mathbb{R}$. By $S(x, r)$ and $B(x, r)$ we denote the sphere and the ball centered at $x \in X$ and radius $r > 0$, respectively. We also write $S(X) = S(0, 1)$ and $B(X) = B(0, 1)$ to denote the unit sphere and the unit ball of X.

Let (Ω, Σ, μ) be a σ-finite complete measure space. By $L_0 = L_0(\Omega)$ we denote the set of all (equivalence classes with respect to the equality μ-a.e. of) measurable extended-real valued functions on Ω. Let $(E, \|\cdot\|_E)$ be a Banach lattice on (Ω, Σ, μ), that is $E \subset L_0$ and if $|x| \leqslant |y|$ μ-a.e. on Ω, $x \in L_0$, $y \in E$ then $x \in E$ and $\|x\|_E \leqslant \|y\|_E$. For any function $x \in L_0$, the support of x is defined by $\operatorname{supp}(x) = \{t \in \Omega : x(t) \neq 0\}$. An element $x \in E$ is called order continuous if for every $0 \leqslant x_n \leqslant |x|$ such that $x_n \downarrow 0$ μ-a.e. it holds $\|x_n\|_E \to 0$.

We say that a Banach lattice $(E, \|\cdot\|_E)$ is *strictly monotone* if for all $x, y \in E$, $\|y\|_E < \|x\|_E$ whenever $0 \leqslant y \leqslant x$ and $x \neq y$.

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3. Steinhaus property in Banach spaces

In this section we provide elaborated proofs of some results obtained in [2].

Theorem 3.1 ([2]Th. 2)**.** *A Banach space $(X, \|\cdot\|)$ has the Steinhaus property if and only if for every $x, y \in S(X)$ and for every $\delta > 0$ there exists $z \in X$ such that $\|z\| < \delta$ and $\max\{\|x+z\|, \|y+z\|\} > 1 > \min\{\|x+z\|, \|y+z\|\}$.*

Proof. Suppose that X has the Steinhaus property. Let $x, y \in S(X)$, $x \neq y$ and $\delta \in (0,1)$. Let $A = \{y, x, 2x, 3x, \ldots\}$. Clearly A is quasi-finite. By the Steinhaus property there is $u \in B((\delta/4)x, \delta/4)$ such that for some $r > 0$, $B(u, r)$ contains exactly one element of A. Since

$$\|x-u\| \leqslant \|x-(\delta/4)x\| + \|(\delta/4)x - u\| < 1,$$

without loss of generality we can assume that $x \in B(u,r)$, $y \notin B(u,r)$ and $r < 1$. We have that $\|u\| \leqslant \|u-(\delta/4)x\| + \|(\delta/4)x\| < \delta/2$. Hence by the triangle inequality

$$1-\delta/2 < \|x-u\| < r \leqslant \|y-u\| < 1+\delta/2. \tag{1}$$

It follows that $r = 1-\epsilon$ for some $\epsilon \in (0, \delta/2)$. Let ρ be any number satisfying

$$0 < \rho < r - \|x-u\|. \tag{2}$$

From (1) we have that $\rho < \delta/2 - \epsilon$. Define $w = y-u$, $v = -(\epsilon+\rho)w/\|w\|$ and $z = -(u+v)$. Clearly $\|v\| = \epsilon+\rho$ and by (1) $\|w\| \geqslant r$. It follows that $\|z\| = \|u+v\| \leqslant \epsilon+\rho+\delta/2 < \delta$ and

$$\|y+z\| = \|y-(u+v)\| = \|w(1+(\epsilon+\rho)/\|w\|)\| = \|w\|+\epsilon+\rho \geqslant r+\epsilon+\rho > 1.$$

Moreover by (2)

$$\|x+z\| = \|x-(u+v)\| \leqslant \|x-u\| + \|v\| < r-\rho+\epsilon+\rho = r+\epsilon = 1.$$

Hence $\max\{\|x+z\|, \|y+z\|\} > 1 > \min\{\|x+z\|, \|y+z\|\}$.

Now assume that for every $x, y \in S(X)$ and for every $\delta > 0$ there exists $z \in X$ such that $\|z\| < \delta$ and $\max\{\|x+z\|, \|y+z\|\} > 1 > \min\{\|x+z\|, \|y+z\|\}$. Let $A \subset X$ be a quasi-finite set. Define

$$G_n = \{x \in X : |A \cap B(x,r)| = n \text{ for some } r > 0\},\ n \in \mathbb{N} \cup \{0\}. \tag{3}$$

Fix $n \in \mathbb{N}$. First we show G_n is open. Let $x \in G_n$, that is there is $r > 0$ such that $|A \cap B(x,r)| = n$. Let $0 < \epsilon < \min\{r - \|x-a\| : a \in A \text{ and } \|x-a\| < r\}$. It follows that $A \cap B(x,r) \subset B(x, r-\epsilon)$. Now for $y \in B(x, \epsilon/2)$ and $a \in A \cap B(x,r)$,

$$\|y-a\| \leqslant \|x-a\| + \|y-x\| < r-\epsilon+\epsilon/2 = r-\epsilon/2.$$

Moreover, for $c \in A \setminus B(x,r)$,

$$\|y-c\| \geqslant |\|y-x\| - \|x-c\|| \geqslant r-\epsilon/2.$$

Hence $|A \cap B(y, r-\epsilon/2)| = n$, that is $y \in G_n$. Hence G_n is open.

Now we show that each G_n is dence in X. Assume towards contradiction that there is $n \in \mathbb{N}$ such that G_n is not dense in X. Hence there is an open ball $B(x_0, r_0)$, $x_0 \in X$, $r_0 > 0$ such that

$$G_n \cap B(x_0, r_0) = \emptyset. \tag{4}$$

By definition of G_n, either $|A \cap B(x_0, r_0)| > n$ or $|A \cap B(x_0, r_0)| < n$. Since A is quasi-finite by taking r_0 large enough, we assume, without loss of generality, that $|A \cap B(x_0, r_0)| > n$.

Define $I_0 = A \cap B(x_0, r_0)$, $J_0 = A \cap S(x_0, r_0)$ and $L_0 = A \setminus (I_0 \cup J_0)$. Clearly $|I_0| + |J_0| > n$. Let $\delta_0 = r_0$.

Suppose that for a positive integer i, for all $j \in \{1, 2, \ldots, i-1\}$, x_j, δ_j, I_j, J_j and L_j are defined and satisfy $|I_j| + |J_j| > n$,

$$x_j \in B(x_0, r_0), \tag{5}$$

$\|x_j - b\| < \|x_j - c\|$ for all $b \in J_j$, $c \in L_j$, and if $I_j \neq \emptyset$ then $\|x_j - a\| < \|x_j - b\|$ for all $a \in I_j$, $b \in J_j$. Clearly, all the assumptions are satisfied if $i = 1$.

Let $D_i = \{m < n : |A \cap B(x_{i-1}, r)| = m$ for some r $> 0\}$, $m_i = \max D_i$ and $r_i = \max\{r > 0 : |A \cap B(x_{i-1}, r)| = m_i\}$. Note that r_i exists because A is quasi-finite. Define $I_i = A \cap B(x_{i-1}, r_i)$, $J_i = A \cap S(x_{i-1}, r_i)$ and $L_i = A \setminus (I_i \cup J_i)$. Clearly $|I_i| = m_i$ and $|I_i| + |J_i| \geqslant n$. Let $k_i = |J_i|$. Since A is quasi-finite we can define a positive number

$$\lambda_i = \min\{\|x_{i-1} - c\| - r_i : c \in L_i\}/2. \tag{6}$$

If $|I_i| + |J_i| = n$ then $|A \cap B(x_{i-1}, r_i + \lambda_i)| = n$. Hence $x_{i-1} \in G_n$ what by (5) contradicts (4) and the proof denseness of G_n is finished.

Since $|J_i| = 1$ implies that $|I_i|+|J_i| = n$, by the above, we may assume that $|J_i| > 1$ and $|I_i|+|J_i| > n$. Assuming that $\min \emptyset = \infty$ we define

$$\gamma_i = \min\{r_i - \|x_{i-1} - a\| : a \in I_i\}/2 \tag{7}$$

and choose a positive number δ_i such that

$$\delta_i < \min\{\delta_{i-1}, \lambda_i, \gamma_i\}/(r_i 2^i). \tag{8}$$

By $|J_i| > 1$ we choose two distinct elements $b_i^{(1)}, b_i^{(2)} \in J_i$. Since $(x_{i-1} - b)/r_i \in S(X)$ for every $b \in J_i$ by the assumption we get that there is $z_i \in X$ with $\|z_i\| < \delta_i$ such that $\|(x_{i-1} - b_i^{(1)})/r_i + z_i\| < 1$ and $\|(x_{i-1} - b_i^{(2)})/r_i + z_i\| > 1$. Defining $x_i = x_{i-1} + r_i z_i$ we get that

$$\|x_i - b_i^{(1)}\| < r_1 \text{ and } \|x_i - b_i^{(2)}\| > r_1.$$

Moreover

$$\|x_i - x_0\| \leqslant \|x_{i-1} - x_0\| + r_i\|z_i\| \leqslant r_1\|z_1\| + r_2\|z_2\| + \ldots + r_i\|z_i\| < r_0 \sum_{j=1}^{i} 2^{-j} < r_0.$$

Hence $x_i \in B(x_0, r_0)$. We have that for all $b \in J_i$ and for every $c \in L_i$,

$$\|x_i - b\| \leqslant \|x_{i-1} - b\| + r_i\|z_i\| < \|x_{i-1} - b\| + \lambda_i \leqslant \|x_{i-1} - b\| + (\|x_{i-1} - c\| - r_i)/2 = (r_i + \|x_{i-1} - c\|)/2$$

and for all $c \in L_i$,

$$\|x_i - c\| = \|x_{i-1} - c - (-r_i z_i)\| > \|x_{i-1} - c\| - \lambda_i \geqslant \|x_{i-1} - c\| - (\|x_{i-1} - c\| - r_i)/2 = (r_i + \|x_{i-1} - c\|)/2.$$

Furthermore, if $I_i \neq \emptyset$ then for all $a \in I_i$,,

$$\|x_i - a\| \leqslant \|x_{i-1} - a\| + r_i\|z_i\| < \|x_{i-1} - a\| + \gamma_i \leqslant \|x_{i-1} - a\| + (r_i - \|x_{i-1} - a\|)/2 = (r_i + \|x_{i-1} - a\|)/2$$

and for all $a \in I_i$ and for every $b \in J_i$,

$$\|x_i - b\| = \|x_{i-1} - b - (-r_i z_i)\| > \|x_{i-1} - b\| - \gamma_i \geqslant \|x_{i-1} - b\| - (r_i - \|x_{i-1} - a\|)/2 = (r_i + \|x_{i-1} - a\|)/2.$$

Now we see that all the assumptions which held true for $i - 1$, hold true for i. By (3.1) we get that $|J_i| < |J_{i-1}|$ if $i \geqslant 2$. Therefore we can repeat the process until a contradiction with non denseness of G_n is found.

Since all the sets G_n, $n \in \mathbb{N}$ are open and dense, by the Baire Category Theorem the set $Y = \cap_{n=1}^{\infty} G_n$ is dense in X. It follows that for all $y \in Y$ and $n \in \mathbb{N}$ there exists an open ball $B(y, r)$, $r > 0$ such that $|A \cap B(y, r)| = n$. Hence X satisfies the Steinhaus property.

□

By using the above characterization we can prove the following result.

Theorem 3.2 ([2] Cor. 3)**.** *Every strictly convex Banach space satisfies the Steinhaus property.*

Proof. Suppose that X is strictly convex. Let $x, y \in S(X)$, $x \neq y$ and $\delta > 0$. Define $z = (\delta/4)x - (\delta/4)y$. Clearly $\|z\| < \delta$. Recall that strict convexity of X gives $\|\alpha x + (1 - \alpha)y\| < 1$ for all $\alpha \in (0, 1)$. Hence by taking $\alpha = 1 - (\delta/4)$, $\|y + z\| < 1$. By the triangle inequality $\|x + z\| \geqslant 1$. If $\|x + z\| = 1$, since $x \neq x + z$, strict convexity of X gives $\|x + (1/2)z\| < 1$. However

$$\|x + (1/2)z\| = \|(1 + \delta/8)x - (\delta/8)y\| \geqslant 1.$$

That is a contradiction. Hence $\|x + z\| > 1$. By Theorem 3.1 we get that X has the Steinhaus property. $\square$

4. Steinhaus property in some Banach lattices

In this section we prove that a large class of Banach lattices over a non-atomic measure space satisfies the Steinhaus property.

Lemma 4.1. *A strictly monotone Banach lattice $(E, \|\cdot\|_E)$ has the Steinhaus property if and only if for every $x, y \in S(E)$, $x, y \geqslant 0$, $x \neq y$ and for every $\delta > 0$ there is $z \in E$ with $\|z\| < \delta$ such that* $\max\{\|x + z\|_E, \|y + z\|_E\} > 1 > \min\{\|x + z\|_E, \|y + z\|_E\}$.

Proof. Let $x, y \in S(E)$, $x \neq y$ and $\delta > 0$. Let $D = \{t \in \Omega : x(t) \neq y(t)\}$. If $\operatorname{sign}(x) \neq \operatorname{sign}(y)$ then, without loss of generality, there is a measurable set $C \subset D$ with positive measure such that $x(t) < 0$, $y(t) \geqslant 0$ and $y(t) - x(t) \geqslant \eta$ for all $t \in C$ and some $\eta > 0$. Let a measurable set $B \subset C$ with $\mu(B) > 0$ and $\beta > 0$ be such that $\beta \leqslant |x(t)|$ for $t \in B$. Let $0 < \gamma \leqslant \min\{\beta, \eta\}$ be so small that $z = (\gamma/2)\chi_B$ has $\|z\| < \delta$. Note that by the choice of β, $0 \leqslant z \leqslant |x|$, hence $z \in E$. Clearly $|x+z| \leqslant |x|$, $|x+z| \neq |x|$ and $|y + z| \geqslant |y|$, $|y + z| \neq |y|$. Since E is strictly monotone we get that $\|x + z\|_E < 1$ and $\|y + z\|_E > 1$.

If $\operatorname{sign}(x) = \operatorname{sign}(y)$ then $|x| \neq |y|$. By the assumption there exists $\tilde{z} \in E$ with $\|\tilde{z}\| < \delta$ such that $\max\{\||x| + \tilde{z}\|_E, \||y| + \tilde{z}\|_E\} > 1 > \min\{\||x| + \tilde{z}\|_E, \||y| + \tilde{z}\|_E\}$. By taking $z = \tilde{z}\operatorname{sign}(x)$ the claim follows. $\square$

Theorem 4.2. *Let $(E, \|\cdot\|_E)$ be a strictly monotone order continuous Banach lattice on a non-atomic measure space. Then $(E, \|\cdot\|_E)$ has the Steinhaus property.*

Proof. We apply Lemma 4.1. Let $x, y \in S(E)$, $x, y \geqslant 0$, $x \neq y$ and $\delta > 0$. Without loss of generality, there is a measurable set D with positive measure and $\eta > 0$ such that $y(t) - x(t) \geqslant \eta$ for $t \in D$. Since E is order continuous and the measure space is non-atomic there is a measurable set $C \subset D$ with $\mu(C) > 0$ such that $0 < \|(x + y)\chi_C\|_E < \delta$. Let $z = -(x + y)\chi_C$. We get that $|x + z| \geqslant x + \eta\chi_C$ and $|y + z| \leqslant y - \eta\chi_C$. Now by applying the strict monotonicity of E the claim follows. $\square$

Strict monotonicity of a Banach lattice over a non-atomic measure space is not sufficient for the Steinhaus property. An example which shows that is the space $(L_1 \cap L_\infty)(\Omega, \Sigma, \mu)$ over a finite measure space such that $\mu(\Omega) > 1$ and equipped with the norm $\|f\| = \|f\|_1 + \|f\|_\infty$. To see that it is enough to take pairwise disjoint measurable sets $A, B, C \subset \Omega$ with positive measure such that

$$(\mu(A) + 1)(1 + \eta) < \mu(B) + \mu(C)$$

for some small $\eta > 0$ and $\mu(B) = \mu(C)$, and define

$$f = \frac{1}{2}\frac{1}{\mu(A) + 1}\chi_A + \frac{1}{2}\frac{1}{\mu(B) + \mu(C)}\chi_{B\cup C}$$

and

$$g = \frac{1}{2}\frac{1}{\mu(A) + 1}\chi_A + \frac{1}{2}\frac{1 - \eta}{\mu(B) + \mu(C)}\chi_B + \frac{1}{2}\frac{1 + \eta}{\mu(B) + \mu(C)}\chi_C$$

and take $\delta \in (0, m)$, where

$$m = \min\left\{\frac{1}{2}\left(\frac{1}{2}\frac{1}{\mu(A) + 1} - \frac{1}{2}\frac{1 + \eta}{\mu(B) + \mu(C)}\right), \frac{1}{2}\frac{1 - \eta}{\mu(B) + \mu(C)}\right\}.$$

Observe that $\|h\| < \delta$ implies $|h| \leqslant \delta$ μ-a.e. on Ω. Hence $f + h \geqslant 0$, $g + h \geqslant 0$, $\|f + h\|_\infty = \|(f + h)\chi_A\|_\infty = \|(g + h)\chi_A\|_\infty = \|g + h\|_\infty$ and $\|f + h\|_1 = \|g + h\|_1$.

An example of an order continuous non strictly monotone Banach lattice over a non-atomic measure space is the space $(L_1 + L_\infty)(\Omega, \Sigma, \mu)$ such that $\mu(\Omega) < \infty$ equipped with the norm

$$\|f\| = \inf\{\max\{\|g\|_1, \|h\|_\infty\} : f = g + h, g \in L_1, h \in L_\infty\}.$$

In view of Theorem 4.2 this space may be considered in order to determine whether the assumption of order continuity itself is also not sufficient for the Steinhaus property. Unfortunately, we were unable to establish whether $(L_1 + L_\infty)$ fails the Steinhaus property or not.

An example of a Banach lattice which fails the Steinhaus property is $L_\infty(\Omega, \Sigma, \mu)$ over a non-atomic measure space. Indeed, it is enough to consider $f = \chi_A + (1/2)\chi_{\Omega\setminus A}$ and $g = \chi_A + (1/4)\chi_{\Omega\setminus A}$, where A is a measurable set with positive measure such that $\mu(\Omega \setminus A) > 0$.

An example of a non strictly convex Banach space which satisfies the Steinhaus property is $L_1(\Omega, \Sigma, \mu)$ over a non-atomic measure space. That is the case since L_1 is a strictly monotone Banach lattice. Note that the sequence space ℓ_1 fails the Steinhaus property. To see that it is enough to consider $x = (1/2, 1/2, 0, 0, \ldots)$ and $y = (2/5, 3/5, 0, 0, \ldots)$.

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