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GROEBNER BASES IN TEACHING COMPUTATIONAL METHODS IN ENGINEERING

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Groebner Bases in Teaching Computational Methods in Engineering

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Abstract. A successful attempt of including Groebner basis method in teaching advanced computational methods in engineering to students at Tennessee Tech University (TTU) is reported. One example of power flow study selected among the student research projects on applying the Groebner basis method to various engineering problems is presented. A general goal is to expand engineering curriculum at TTU and to expose engineering students to areas of advanced mathematics not traditionally included in engineering curricula.

Keywords: Groebner basis, algebraic geometry, power flow analysis, material matrix, geometrically nonlinear plates

1 Introduction

With significant advances in Computer Algebra Systems like Maple, Mathematica, Reduce, CoCoCA, Macaulay2, etc., the Groebner basis theory introduced by Buchberger [1] in 1965 has attracted more and more attention in various scientific fields. It has become an effective mathematical tool and a feasible option in modeling and simulation for problems that have a nonlinear and/or highly coupled nature. However, until recently, this advanced modern computational method has had a limited impact on the engineering practice. As a result of the authors' research collaboration in the area of algebraic geometry [2], the first author was able to develop a graduate one-semester course "Advanced Computational Methods in Engineering" in which doctoral, master, and Fast-Track undergraduate engineering students learn the method of Groebner bases and then apply it to solving various engineering problems. At that time, no such graduate course was offered in Mathematics or Engineering Departments at TTU that provided this knowledge and its applications in engineering. Since 2006, when the course was introduced, approximately thirty students have completed it. The course syllabus is presented in Appendix A. Due to the page limit to this note, only one example of the student research projects using Groebner bases in the areas of electrical, mechanical, civil, and engineering mechanics is reported in Section 3. It appeared in a conference paper at the 41th North American Power Symposium [5]. All projects have been done with Maple Computer Algebra System. The authors' project discussed in Section 4 is currently under way. Its goal is to create a Maple package for teaching "Advanced Mechanics of Materials" and "Reinforced Composite Materials". The package uses Groebner bases. When using the package, students will not only better understand material properties taught in the courses, but, equally importantly, they will learn how to describe these properties mathematically. Thus, a general goal of teaching these methods is to expand engineering curriculum at TTU and expose students to areas of advanced mathematics not traditionally included in engineering curricula.

2 Course Development

The course CEE 7100: Advanced Computational Methods in Engineering has two parts: in the first part, students learn basic concepts of algebraic geometry, affine spaces and varieties, and acquire a theoretical background in Groebner bases including Buchberger's algorithm for computation of Groebner bases. In the second part, using Maple, students apply Groebner bases to various problems related to their research in engineering practice. For the course syllabus, see Appendix A.

3 An Example of a Student Project: Power Flow Study

All the student course projects show that in the one semester course students are able to 1. learn the Groebner basis method and 2. demonstrate that they can now solve various nonlinear engineering problems related to their research areas. The Maple Computer Algebra System has been used in all course projects to implement the conversion of a set of nonlinear algebraic equations stemming from an analysis of the problems into an equivalent set of uncoupled polynomial algebraic equations (the reduced Groebner basis). The four best research projects were presented by the students at the 43rd and 44th Annual Technical Meeting Society of Engineering Science in 2006 and 2007, and appeared in the conference proceedings. As an example, the project by Jiaxin Ning on the analysis of power flow in the area of electrical engineering is presented here briefly.

The problem arises in the analysis of a three-bus power flow system (see [5]). The diagram of the system is shown in Fig. 1. Bus 1 is a slack bus, where the voltage angle and magnitude of the bus are known; bus 2 is a PV bus, where the voltage magnitude of the bus and the injected real power are known; bus 3 is a PQ bus, where the real and reactive power are known. The system variables are the voltages on bus 2 and bus 3 in the form of power flow equations with a_2, a_3, b_2, and b_3 being unknown constants to be found. As a result, one has to solve

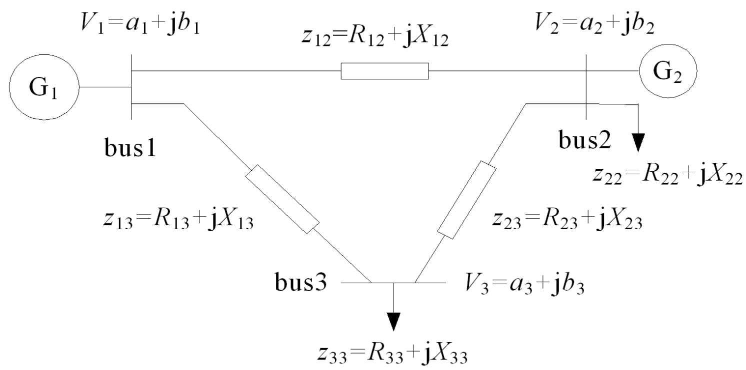

Fig. 1. Diagram of a three-bus power system

the following system of polynomial equations:

$$P_2 = \sum_{j=1}^{3} a_2(G_{2j}a_j - B_{2j}b_j) + \sum_{j=1}^{3} b_2(G_{2j}b_j + B_{2j}a_j), \tag{1}$$

$$V_2 = a_2^2 + b_2^2, \tag{2}$$

$$P_3 = \sum_{j=1}^{3} a_3(G_{3j}a_j - B_{3j}b_j) + \sum_{j=1}^{3} b_3(G_{3j}b_j + B_{3j}a_j), \tag{3}$$

$$Q_3 = \sum_{j=1}^{3} b_3(G_{3j}a_j - B_{3j}b_j) - \sum_{j=1}^{3} a_3(G_{3j}b_j + B_{3j}a_j). \tag{4}$$

The MAPLE Groebner basis module was used to un-couple equations (1)–(4) into one fifth order primary equation containing only b_3 and three other equations containing b_3 and one of a_2, b_2, and a_3, respectively. These equations (which contain all the system parameters in the symbolic form) are not shown for the sake of brevity. The admittance of the system and other known variables are listed in (5) and (6). With the input data given in (5) and (6), Table 1 compares results obtained from the Newton-Raphson (NR) method commonly used in the power flow study with results obtained from the Groebner basis method. Only a small amount of computer time was needed to generate these results despite the large number of extraneous solutions inherent in the process. Since the solution of the system by the NR method highly depends upon the selection of initial values that are usually unknown, especially during transients of the system, the potential benefit of Groebner basis method is twofold: 1. it allows to compute the solution of the power flow, and 2. it allows one to analyze the relationship between the initial conditions of the system and its convergent property which is very important in any dynamic system simulation.

$$G = \begin{bmatrix} 6.25 & -5 & -1.25 \\ -5 & 6.6667 & -1.6667 \\ -1.25 & -1.6667 & 2.9167 \end{bmatrix}, \quad B = \begin{bmatrix} -18.75 & 15 & 3.75 \\ 15 & -20 & 5 \\ 3.75 & 5 & -8.75 \end{bmatrix} \tag{5}$$

$$a_1 = 1.05, b_1 = 0, V_2 = 1.1, P_2 = 0.517, P_3 = 0.942, Q_3 = 0.19 \tag{6}$$

Table 1. Results from Newton-Raphson method and Groebner Basis technique

	NR		GB Solution 1		GB Solution 2	
Bus	a	b	a	b	a	b
1	1.05	0	1.05	0	1.05	0
2	1.0986	0.0553	1.096	0.0553	1.0987	−0.0538
3	1.0458	0.0606	1.0459	0.0606	0.032	−0.0922

4 A New Package for Teaching Engineering Mechanics with Maple

In this section we report on a new Maple package called `MaterialConstants` [3] to display a material constants matrix with the following symmetries: monoclinic, orthotropic, transversely isotropic, isotropic, cubic isotropic, tetragonal, trigonal, trigonal-hexagonal, and other. The software applies appropriate symmetries T_σ and T_ϵ^{-1} to a symmetric 6×6 material constants matrix C, that is, $C \to T_\sigma C T_\epsilon^{-1}$ where T_σ and T_ϵ are maps $\mathbb{R}^6 \to \mathbb{R}^6$ which are applied to the stress array σ and the strain array ϵ, respectively. The maps are determined by the required material symmetry transformations. Students can easily see which constants are non-zero and which are independent for each type of symmetry. When finding the transformed matrix C, Groebner basis are computed twice with respect to an appropriate monomial order. For an executable Maple worksheet and its `.pdf` version, visit `http://math.tntech.edu/rafal/cliff15/`.

5 Conclusions

The foregoing has presented the example of the application of Groebner basis technique to the analysis of power system in electrical engineering. The example has demonstrated the capability of the MAPLE Groebner basis module to generate a Groebner basis using a standard PC if a number of unknowns is small. For many problems in engineering mechanics, low order Rayleigh/Ritz, Galerkin, and similar approximate methods of weighted residuals tend to produce correspondingly low order systems of polynomial algebraic equations. In such cases, the Groebner basis approach is of great value in generating closed form solutions that cannot be found numerically. In that sense there is a useful connection between methods of weighted residuals and Groebner basis technique.

A CEE 7100 Course Syllabus

I. Catalog Data:
CEE 7100: Advanced Computational methods in Engineering Lecture 3, Credit 3.

II. Prerequisites:
CEE/ME 6930 and an additional graduate level course in engineering mechanics or consent of instructor.

III. Course Description:
Basic concepts of real algebraic geometry, Groebner bases theory and Buchberger's algorithm, computational applications to solving systems of equations using the Groebner bases method, particularly on the relevance of mechanics and engineering problems in vibration and buckling eigenvalue problems, Lagrange multiplier problems, energy method and geometric nonlinear problems in plates and shells, mesh generation for composites in finite element analysis, geometric description of robots, and inverse kinematics and motion planning. Necessary and required topics will also include: polynomial rings and affine space, projective and affine varieties, ideals.

IV. Course Objectives:
The purpose of this introductory course is to give graduate level students a better understanding of the mathematical tools available for engineering research, allow them to "put their foot in the door" in the field of advanced computational analysis, and provide a starting point for future endeavors in advanced level academic research. In the one semester course students are able to 1. learn the Groebner basis method and Buchberger's algorithm to compute Groebner bases and 2. demonstrate that they can use the method to solve various nonlinear engineering problems related to their research areas and **show how these relatively new mathematical tools can be used to solve problems in current engineering applications and research**.

V. Topics to be Covered:

Part 1. Introduction to Necessary Concepts of the Groebner Basis Theory

- Basic concepts of algebraic geometry
- Polynomial rings, affine space, and ideals
- Projective and affine varieties
- The Hilbert basis theorem and Groebner Bases
- Properties of Groebner bases
- Buchberger's algorithm
- Groebner bases method and applications
- First applications - curves, surfaces, ruled surfaces, tangent surfaces, Bézier cubic and shape functions, solve systems of polynomial equations

- Improvements on Buchberger's algorithm (optional)

Part 2. Groebner Basis Approach to Engineering Applications
- Parallel lines, curves and surfaces applied to shell structures
- Solve nonlinear systems of polynomial and rational equations
- Vibration and buckling eigenvalue problems
- Lagrange multipliers problems
- Nonlinear problems in mechanics with energy method
- Pre-process for finite element analysis modeling and mesh generation for composite shells
- Geometric description of robots
- The forward kinematics, inverse kinematics and motion planning (optional)
- Additional applications in statics, dynamic, and stability problems.

VI. Suggested Texts and References:
- *Ideals, Varieties, and Algorithms—An Introduction to Computational Algebraic Geometry and Commutative Algebra*, D. Cox, J. Little, and D. O'Shea, 3rd Ed., 2007, Springer. ISBN: 978-0-387-35650-1
- *A Primer of Abstract Mathematics*, R. B. Ash, The Mathematical Association of America
- *Geometric Fundamentals of Robotics*, J. M. Selig, 2nd Ed., Springer
- *Stresses in Plates and Shells*, A. C. Ugural, 2nd Ed., McGraw-Hill
- *Theory of Elasticity*, S. P. Timoshenko and J. N. Goodier, McGraw-Hill
- *Foundation of Solid Mechanics*, Y. C. Yung, Prentice-Hall, N. J.

VII. Computer Software and Programming Language That May be Used:

Maple package, Fortran or C, finite element analysis package (ANSYS or ABAQUS)

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