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REPRESENTATIONS AND CHARACTERS OF SALINGAROS' VEE GROUPS OF LOW ORDER

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Abstract. We study irreducible representations and characters of Salingaros' vee groups of orders 4, 8, and 16 as 2-groups of exponent 4. In particular, we construct complex irreducible group modules and explicit representations of these groups. We prove a theorem regarding the number of conjugacy classes and the number of inequivalent irreducible representations of degree one and two. We show how to decompose a complex group algebra into irreducible submodules in accordance with Maschke's Theorem. We formulate two algorithms for finding bases for these submodules which rely on the Groebner basis methods. In the end, we provide the character tables of these groups.

1. Introduction

In a series of papers, Salingaros [18–20] studied a connection between finite groups and Clifford algebras [14]. He described five types of finite groups that are related to real Clifford algebras $C\ell_{p,q}$ and spinors. He called these groups 'vee groups' and the five classes he labelled as N_{odd}, N_{even}, Ω_{odd}, Ω_{even}, and S_k. In 1988 Shaw [21] studied a group of order 128 which he associated with the Clifford algebra $C\ell_{0,7}$, while O'Brien and Slattery [15] investigated the structure of finite groups associated with Clifford algebras of signature $(0, d), d \equiv 3 \pmod 4$. Later, Albuquerque and Majid [5] viewed Clifford algebras as group algebras of $\mathbb{Z}_2^n$ twisted by a cocycle. They obtained periodicity properties of the Clifford algebras and presented a new approach to their spinor representations through the twisted group algebra. Matrix

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representations of generalized Clifford algebras viewed as twisted group rings $\mathbb{Z}_2^n$ were also studied by Caenepeel and Van Oystaeyen [7]. More recently, the 'vee groups' appeared in [2–4] where (while denoted as $G_{p,q}$) their transitive action on complete sets of mutually annihilating primitive idempotents was studied. Using the normal stabilizer subgroup $G_{p,q}(f)$ of a primitive idempotent f, left transversals, spinor bases, and maps between spinor spaces for different orthogonal idempotents f_i summing up to 1 were described while the finite stabilizer groups according to the signature in simple and semisimple cases were classified. Most recently, Salingaros' 'vee groups' have appeared in Varlamov [22] in his study of CPT groups for spinor fields in de Sitter and anti-de Sitter spaces.

In this paper we apply methods of representation theory of finite groups and their characters [12, 16, 17] to construct irreducible representations of Salingaros' 'vee groups' of orders $4, 8$, and 16. As a byproduct we obtain their character tables. Throughout, G always denotes a finite group and any action of G on a G-module is a left action.

To establish notation and the background for later computations, we recall basic results of the theory of representations and characters of finite groups. [12]

Theorem 1. ***(Maschke)*** *Let G be a finite group and let V be a nonzero G-module. Then $V = W^{(1)} \oplus W^{(2)} \oplus \cdots \oplus W^{(k)}$, where each $W^{(i)}$ is an irreducible G-submodule of V.*

As a result of Maschke's theorem, matrices of any representation can be written in the block-diagonal form once a suitable basis in the representation module is chosen. Thus, a representation is irreducible or it is *completely reducible* meaning that it can be written as a direct sum of irreducible representations.

Corollary 1. *Let G be a finite group and let X be a matrix representation of G of dimension $d > 0$. Then there is a fixed matrix T such that every matrix $X(g)$, $g \in G$, has the diagonal form*

$$TX(g)T^{-1} = \operatorname{diag}\left(X^{(1)}(g), X^{(2)}(g), \ldots, X^{(k)}(g)\right) \tag{1}$$

Given a reducible representation, it is interesting to derive a procedure to find a basis in the module to completely reduce it per Maschke's Theorem. Since later we will work exclusively with group algebras, we recall the following result.

Let V be a complex vector space with an inner product $\langle \cdot, \cdot \rangle$ and let $W^{\perp}$ denote the orthogonal complement of a subspace W of V. It is always true that $V = W \oplus W^{\perp}$ and that this decomposition is G-invariant as long as the inner product is G-invariant, that is, $\langle g\mathbf{u}, g\mathbf{v} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle$ for any $g \in G$ and $\mathbf{u}, \mathbf{v} \in V$. This leads to the following known result [17].

Proposition 1. *Let V be a G-module, W a submodule, and $\langle \cdot, \cdot \rangle$ an inner product invariant under the action of G. Then $W^{\perp}$ is also a G-submodule.*

Using the G-invariance of the orthogonal complement allows one to find a decomposition of any representation module into irreducibles. Following [3, 4], we denote Salingaros' 'vee group' related to the universal Clifford algebra $C\ell_{p,q}$ as $G_{p,q}$. Let $C\ell_{p,q}$ be a real universal Clifford algebra with Grassmann basis $\mathcal{B}$ sorted by the admissible order `InvLex` [2]. We begin by recalling an informal definition of Salingaros' vee group [20].

Definition 1. *A vee group $G_{p,q}$ is defined as the set $G_{p,q} = \{\pm m \mid m \in \mathcal{B}\}$ in $C\ell_{p,q}$ together with the Clifford product as the group binary operation.*

This finite group is of order $|G_{p,q}| = 2^{1+p+q}$ and its commutator subgroup $G'_{p,q} = \{\pm 1\}$. The following result is well known. [12, 17]

Theorem 2. *Let G be a finite group. The number of degree 1 representations of G is $[G : G']$ where G' is the commutator subgroup of G.*

Example 1. *Let $G_{p,q}$ be the Salingaros' vee group of the Clifford algebra $C\ell_{p,q}$ [18–20]. The commutator subgroup $G'_{p,q}$ is $\{1\}$ when $p+q=1$ and it is $\{1,-1\}$ when $p+q \geq 2$. Thus, $G_{p,q}/G'_{p,q} \cong (\mathbb{Z}_2)^n$ where $n = p+q \geq 2$. In the special case $n = p+q = 1$ and the signature $(1,0)$, the group $G_{1,0}/G'_{1,0} \cong G_{1,0} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, whereas in the signature $(0,1)$, the group $G_{0,1}/G'_{0,1} \cong G_{0,1} \cong \mathbb{Z}_4$.*

For the remainder of the current section, we recall basic results from the theory of representations and characters that will be needed in the later sections.

Let $x, y \in G$. Then, x is *conjugate* to y in G if $y = g^{-1}xg$ for some $g \in G$. The set of all elements conjugate to $x \in G$ is called the *conjugacy class* of x in G and it is denoted by x^G. Thus, $x^G = \{g^{-1}xg : \ g \in G\}$. Recall that conjugacy is an equivalence relation, and that the conjugacy classes are the equivalence classes. Thus, every group can be represented as a disjoint union (a partition) of conjugacy classes.

The size of each class is the index of the centralizer of any representative of the conjugacy class in G. The *centralizer* of $x \in G$, written $C_G(x)$, is the set of elements of G which commute with x, i.e., $C_G(x) = \{g \in G : xg = gx\}$.

Theorem 3. *Let $x \in G$. Then the size of the conjugacy class x^G is given by*

$$|x^G| = [G : C_G(x)] = |G|/|C_G(x)|. \tag{2}$$

In particular, the number of elements of a conjugacy class is a divisor of the order of the group.

A proof of the theorem is straightforward [12]. To illustrate the theorem, we recall the conjugacy classes of the dihedral group D_{2n}. [12]

Example 2. *Consider the dihedral group* $D_{2n} = \langle r, p : r^n = p^2 = (rp)^2 = e \rangle$ *of order* $2n$. *When* n *is odd,* D_{2n} *has* $\frac{1}{2}(n+3)$ *conjugacy classes. Namely,*

$$\{e\},\ \{r, r^{-1}\},\ \ldots,\ \{r^{(n-1)/2}, r^{-(n-1)/2}\},\ \{p, rp, \ldots, r^{n-1}p\}. \tag{3}$$

When n *is even and* $n = 2m$ *for some positive integer* m *then* D_{2n} *has* $m+3$ *conjugacy classes:*

$$\begin{gathered} \{e\},\ \{r^m\},\ \{r, r^{-1}\},\ \ldots,\ \{r^{m-1}, r^{-(m-1)}\}, \\ \{r^{2j}p : 0 \le j \le m-1\},\ \{r^{2j+1}p : 0 \le j \le m-1\}. \end{gathered} \tag{4}$$

An important relation between characters, conjugacy classes, and equivalent representations is the following standard result [12, 17].

Proposition 2. *Let* X *be a matrix representation of a group* G *of degree* d *with character* χ. *(i)* $\chi(1) = d$, *(ii) If* g, h *belong to a conjugacy class* K *in* G, *then* $\chi(g) = \chi(h)$. *(iii) If* Y *is a representation of* G *with character* ψ, *and if* $X \cong Y$ *then* $\chi(g) = \psi(g)$ *for all* $g \in G$.

A proof of this proposition can be easily given, see [17].

Let $X(g)$, $g \in G$, be a matrix representation, that is, a homomorphism $G \to \mathrm{GL}(n, \mathbb{C})$ (in this paper we always consider representations over the complex field). Then the *character* χ of X is the map $G \xrightarrow{\mathrm{tr}\, X} \mathbb{C}$ such that $\chi(g) = \mathrm{tr}\, X(g)$ where tr denotes the trace of a matrix. If V is a G-module, then its *character* is the character of a matrix representation X on V.

Consider a character χ of a group G as a row vector with complex entries

$$\chi = (\chi(g_1), \chi(g_2), \ldots, \chi(g_n)) \tag{5}$$

where $G = \{g_1, g_2, \ldots, g_n\}$. Inner product of characters can be defined as follows.

Definition 2. *Let* χ *and* ψ *be two characters of a group* G. *The inner product of* χ *and* ψ *is defined as the following sum over* G*:*

$$\langle \chi, \psi \rangle = \frac{1}{|G|} \sum_{g \in G} \chi(g) \overline{\psi(g)}. \tag{6}$$

Based on the above definition, it may be impossible to find the inner product for an arbitrary field because it may lack the conjugation operation. So, the following proposition gives an equivalent form of the inner product which can be used for any field. Also, since all elements in the same conjugacy class have the same character, the formula for the inner product can be further simplified.

Proposition 3. *Let G have k conjugacy classes with representatives $g_1, g_2, \ldots, g_k$. Also, let χ and ψ be some characters of G. Then, $\langle \chi, \psi \rangle = \langle \psi, \chi \rangle$, and*

$$\langle \chi, \psi \rangle = \frac{1}{|G|} \sum_{g \in G} \chi(g)\psi(g^{-1}) = \sum_{i=1}^{k} \frac{\chi(g_i)\overline{\psi(g_i)}}{|C_G(g_i)|} \tag{7}$$

Theorem 4. *Let χ and ϕ be irreducible characters of a group G. The characters are orthonormal with respect to the inner product, i.e., $\langle \chi, \psi \rangle = \delta_{\chi,\psi}$.*

As a consequence of the above theorem, several results can be stated in relation to representations, irreducibility, etc. The following theorem will be used extensively in finding irreducible characters of certain groups in Section 2.

Theorem 5. *Let X be a representation of G with character χ, and $X \cong m_1 X^{(1)} \oplus m_2 X^{(2)} \oplus \cdots \oplus m_k X^{(k)}$, where the $X^{(i)}$ are pairwise inequivalent irreducibles with characters $\chi^{(i)}$ and multiplicities m_i.*

1. $\chi = m_1 \chi^{(1)} \oplus m_2 \chi^{(2)} \oplus \cdots \oplus m_k \chi^{(k)}$.
2. $\langle \chi, \chi^{(i)} \rangle = m_i$ *for all* i.
3. $\langle \chi, \chi \rangle = {m_1}^2 + {m_2}^2 + \cdots + {m_k}^2$.
4. χ *is irreducible if and only if* $\langle \chi, \chi \rangle = 1$.
5. *Let Y be another representation of G with character ψ. Then $X \cong Y$ if and only if $\chi(g) = \psi(g)$ for all $g \in G$.*

Observe that pairwise inequivalent irreducible G-modules give pairwise inequivalent irreducible representations. Then, Maschke's theorem implies that the group algebra $\mathbb{C}[G]$ can be written as $\mathbb{C}[G] = \oplus_i m_i V^{(i)}$ where m_i is the multiplicity of $V^{(i)}$ stating how many times $V^{(i)}$ appears in the decomposition.

Proposition 4. *Let G be a finite group and consider a decomposition of its group algebra $\mathbb{C}[G] = \oplus_i m_i V^{(i)}$ where the $V^{(i)}$ form a complete list of pairwise inequivalent irreducible G-modules. Then,*

1. $m_i = \dim V^{(i)}$,
2. $\sum_i (\dim V^{(i)})^2 = |G|$, *and*
3. *The number of $V^{(i)}$ equals the number of conjugacy classes of G.*

Example 3. *Let* $G = S_n$. *Of course it is well known that the number of conjugacy classes for any* S_n *equals the number of partitions of* n. *Furthermore, each class consists of permutations having the same cycle structure because the action of conjugation preserves the cycle structure.*

Example 4. *Let* $G = D_{2n}$. *The number of conjugacy classes for the dihedral group was discussed in Example 2. For example,* D_6 *and* D_8 *have three and five conjugacy classes, respectively. Thus,*

$$\text{Conjugacy classes of } D_6: \quad \{e\},\ \{r, r^2\},\ \{p, rp, r^2p\}, \tag{8}$$

$$\text{Conjugacy classes of } D_8: \quad \{e\},\ \{r^2\},\ \{r, r^3\},\ \{p, r^2p\},\ \{rp, r^3p\}. \tag{9}$$

Example 5. *Conjugacy classes of the quaternionic (or, dicyclic) group* Q_{2n} *which appears, along with the dihedral group* D_{2n}*, among the Salingaros' vee groups [18–20], can be found by hand (for low orders) or by using a computer software such as the Maple package [1]. For example, the group* $Q_8 = \{\pm\mathbf{1}, \pm\mathbf{i}, \pm\mathbf{j}, \pm\mathbf{k}\}$ *where* $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -\mathbf{1}$*, has the following five conjugacy classes:*

$$\{\mathbf{1}\},\ \{-\mathbf{1}\},\ \{\mathbf{i}, -\mathbf{i}\},\ \{\mathbf{j}, -\mathbf{j}\},\ \{\mathbf{k}, -\mathbf{k}\}. \tag{10}$$

The orthogonality relations of the first kind stated in Theorem 4 are complemented by the so called *orthogonality relations of the second kind*. While the relations of the first kind refer to the rows of the character table, the relations of the second kind refer to the columns.

Theorem 6. *Let* K, L *be conjugacy classes of* G. *Then* $\sum_\chi \chi_K \bar{\chi}_L = \frac{|G|}{|K|}\delta_{K,L}$*, where the sum is over all irreducible characters of* G.

In Section 2, we will compute all irreducible representations and characters of all Salingaros' vee groups of orders 4, 8 and 16, and verify Theorems 4 and 6.

2. Salingaros' vee groups

In this section, we state the definition of Salingaros' vee groups and discuss some of their properties. Then, Salingaros' classification of these groups will be reviewed. Furthermore, irreducible representations and their character tables will be presented for a few sample groups.

2.1. General definitions and properties

The vee groups were introduced by Salingaros in [18–20]. They were more recently studied in [2–4, 22] where they were denoted as $G_{p,q}$. In particular, these groups are central extensions of extra-special 2-groups. [6, 9–11, 13, 22]

Definition 3. *Let $C\ell_{p,q}$ be the real Clifford algebra of a non-degenerate quadratic form with signature (p, q) and let $\mathcal{B} = \{e_{\underline{i}} \mid 0 \le |\underline{i}| \le n\}$ be a basis for $C\ell_{p,q}$ consisting of basis monomials $e_{\underline{i}} = e_{i_1}e_{i_2}\cdots e_{i_k}$, $i_1 < i_2 < \cdots < i_k$, for $0 \le k \le n$ where $n = p+q$. The 1-vector generators e_i, $1 \le i \le n$, satisfy the following relations:*

$$e_i^2 = \begin{cases} 1 & \text{if } 1 \le i \le p, \\ -1 & \text{if } p+1 \le i \le n, \end{cases} \quad \text{and} \quad e_ie_j = -e_je_i \text{ for } i \ne j.$$

The Salingaros' vee group $G_{p,q} \subset C\ell_{p,q}$ is the set $G_{p,q} = \{\pm e_{\underline{i}} \mid e_{\underline{i}} \in \mathcal{B}\}$ with the Clifford algebra multiplication as the group binary operation. Thus, $|G_{p,q}| = 2{\cdot}2^{p+q} = 2^{n+1}$.

Notice that $G_{p,q}$ may be presented as follows:

$$G_{p,q} = \langle -1, e_1, \ldots, e_n \mid e_ie_j = -e_je_i \text{ for } i \ne j \text{ and } e_i^2 = \pm 1\rangle, \tag{11}$$

where $e_i^2 = 1$ for $1 \le i \le p$ and $e_i^2 = -1$ for $p+1 \le i \le n = p+q$. In the following, the elements $e_{\underline{i}} = e_{i_1}e_{i_2}\cdots e_{i_k}$ will be denoted for short as $e_{i_1i_2\cdots i_k}$ for $k \ge 1$ while $e_\emptyset$ will be denoted as 1, the identity element of $G_{p,q}$ (and $C\ell_{p,q}$).

For the properties of the prime power groups we refer to [10, 13, 16].

Theorem 7. ***(Cauchy)*** *If G is a finite group whose order is divisible by a prime p then G contains an element of order p.*

Since all Salingaros' vee groups are of order 2^n for some positive integer n, there are elements of order 2 in $G_{p,q}$. In fact, as it will be seen later, any element in $G_{p,q}$ is of order 1, 2, or 4 only.

Let $G = G_{p,q}$. Since each $C_{G_{p,q}}(x_i)$ is a proper subgroup of $G_{p,q}$ for $x_i \notin Z(G_{p,q})$ and $G_{p,q}$ is a 2-group, Lagrange's theorem gives that $[G_{p,q} : C_{G_{p,q}}(x_i)]$ is a divisor of $|G_{p,q}|$, hence it is a power of 2. This implies that $2 \mid |Z(G_{p,q})|$. Thus, $Z(G_{p,q}) \ne \{1\}$, which gives the following result.

Lemma 1. *The center of any Salingaros' vee group is non trivial and it is of order 2^n for some $n \ge 1$.*

In fact, from the structure theorem of Clifford algebras (see [2, 3] and references therein) one can learn that

$$Z(C\ell_{p,q}) = \begin{cases} \{1\} & \text{if } p+q \text{ is even;} \\ \{1, \beta\} & \text{if } p+q \text{ is odd,} \end{cases} \tag{12}$$

where $\beta = e_1 e_2 \cdots e_n$, $n = p+q$, is the unit pseudoscalar in $C\ell_{p,q}$. This leads to the following conclusion (see also [22]).

Theorem 8. *Let $G_{p,q} \subset C\ell_{p,q}$. Then,*

$$Z(G_{p,q}) = \begin{cases} \{\pm 1\} \cong \mathbb{Z}_2 & \textit{if } p-q \equiv 0,2,4,6 \pmod 8; \\ \{\pm 1, \pm\beta\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2 & \textit{if } p-q \equiv 1,5 \pmod 8; \\ \{\pm 1, \pm\beta\} \cong \mathbb{Z}_4 & \textit{if } p-q \equiv 3,7 \pmod 8). \end{cases} \tag{13}$$

The following result implies that the vee groups of order 2^2 are abelian. For the proof of this proposition, see [16].

Proposition 5. *If p is a prime, then every group G of order p^2 is abelian.*

It is worth to know the order relation of the normal subgroups of the Salingaros' vee groups.

Proposition 6. *If a group G is of order $|G| = p^n$, then G has a normal subgroup of order p^m for every $m \leq n$.*

So, this result tells that $G_{p,q}$ of order 2^{p+q+1} has a normal subgroup of order 2^m for any $m \leq p+q+1$, which implies that $G_{p,q}$ are not simple groups.

2.2. Conjugacy classes

In this section we discuss the conjugacy classes of $G_{p,q}$ using Theorem 8. It is convenient to separately address the two cases when $n = p+q$ is odd or even.

Suppose that n is even. Then, $Z(G_{p,q}) = \{\pm 1\}$ (see Theorem 8) and so $\{1\}$ and $\{-1\}$ are the only conjugacy classes in $G_{p,q}$ with a single element. This is because the unit pseudoscalar β is not in the center. All other classes always have two elements $\{g, -g\}$ for any non central group element g. Thus, the number of conjugacy classes is $N = 1 + 2^{p+q}$ and the classes are:

$$\{1\},\ \{-1\},\ \{e_1, -e_1\},\ \ldots,\ \{\beta, -\beta\} \tag{14}$$

Now consider the second case when n is odd. The center $Z(G_{p,q})$ has four elements $\{\pm 1, \pm\beta\}$. Hence, $\{1\}$, $\{-1\}$, $\{\beta\}$, and $\{-\beta\}$ are the only classes with a single element whereas all other classes have two elements $\{g, -g\}$ for any non central element g. Thus, the number of conjugacy classes is $N = 2 + 2^{p+q}$ and the classes are:

$$\{1\},\ \{-1\},\ \{\beta\},\ \{\beta\},\ \{e_1, -e_1\},\ \ldots,\ \{e_{12\ldots(n-1)}, -e_{12\ldots(n-1)}\}, \tag{15}$$

The above results can be given as the following theorem.

Theorem 9. *Let N be the number of conjugacy classes in $G_{p,q}$. Then,*

$$N = \begin{cases} 1 + 2^{p+q} & \text{if } p+q \text{ is even;} \\ 2 + 2^{p+q} & \text{if } p+q \text{ is odd.} \end{cases} \tag{16}$$

Proof. Note that any two elements $\tau, g \in G_{p,q}$ are basis monomials from $C\ell_{p,q}$ which either commute $\tau g = g\tau$ or anticommute $\tau g = -g\tau$. If $\tau \in Z(G_{p,q})$ then $g\tau g^{-1} = \tau$ for all $g \in G_{p,q}$. So $\tau^{G_{p,q}} = \{\tau\}$ for all $\tau \in Z(G_{p,q})$. If $\tau \notin Z(G_{p,q})$, then there exists $g \in G_{p,q}$ such that $\tau g = -g\tau$, i.e., $g\tau g^{-1} = -\tau$. Hence, $\tau^{G_{p,q}} = \{\tau, -\tau\}$ for all $\tau \notin Z(G_{p,q})$. From Lemma 8, the number of elements in $Z(G_{p,q})$ for even and odd cases are known. Then, the formulas for N follow immediately. ■

Salingaros lists five classes of vee groups in [18] and references therein. He denotes these groups as: N_{2k-1}, N_{2k}, Ω_{2k-1}, Ω_{2k}, S_k. The groups N_{2k-1} and N_{2k} are included in the Clifford algebras $C\ell_{p,q}$ when $p+q$ is even, whereas Ω_{2k-1}, Ω_{2k}, and S_k are included in the Clifford algebras $C\ell_{p,q}$ when $p+q$ is odd. In particular, the groups S_k are included in those Clifford algebras in which the unit pseudoscalar β squares to -1 whereas Ω_{2k-1} and Ω_{2k} are included in those semisimple algebras in which β squares to 1. The basic information about these groups is summarized in Table 1.

One other distinguishing feature of these groups is their order structure which is different from one class to another. Knowing the order structure of $G_{p,q}$ allows one to determine its class. For example, the first few vee groups corresponding to the Clifford algebras in dimensions one, two and three, are:

Groups of order 4: $G_{1,0} = D_4, \quad G_{0,1} = \mathbb{Z}_4,$

Groups of order 8: $G_{2,0} = D_8 = N_1, \quad G_{1,1} = D_4 = N_1, \quad G_{0,2} = Q_8 = N_2,$

Groups of order 16: $G_{3,0} = S_1, \quad G_{2,1} = \Omega_1, \quad G_{1,2} = S_1, \quad G_{0,3} = \Omega_2.$

where D_8 is the dihedral group of a square from Example 4 whereas Q_8 is the quaternionic group from Example 10.

Tab. 1: Vee groups $G_{p,q}$ in Clifford algebras $C\ell_{p,q}$

Group	Center	Group order	$\dim_{\mathbb{R}} C\ell_{p,q}$
N_{2k-1}	$\mathbb{Z}_2$	2^{2k+1}	2^{2k}
N_{2k}	$\mathbb{Z}_2$	2^{2k+1}	2^{2k}
Ω_{2k-1}	$\mathbb{Z}_2 \times \mathbb{Z}_2$	2^{2k+2}	2^{2k+1}
Ω_{2k}	$\mathbb{Z}_2 \times \mathbb{Z}_2$	2^{2k+2}	2^{2k+1}
S_k	$\mathbb{Z}_4$	2^{2k+2}	2^{2k+1}

Definition 4. *The order structure of $G_{p,q}$ is a 3-tuple (n_1, n_2, n_3) of nonnegative integers where n_1, n_2 and n_3 give the number of elements in $G_{p,q}$ of order one, two and four, respectively.*

The following theorem gives the number of inequivalent representations of degree one of the group $G_{p,q}$.

Theorem 10. *Let M be the number of inequivalent representations of degree one of $G_{p,q}$. Then,*

$$M = \begin{cases} 2 \cdot 2^{p+q} = 4 & \text{if } p+q = 1; \\ 2^{p+q} & \text{if } p+q \geq 2. \end{cases} \tag{17}$$

Proof. From Theorem 2, the number of degree one representations of $G_{p,q}$ is the index of its commutator subgroup $[G_{p,q} : G'_{p,q}]$. When $p+q=1$, the commutator subgroup $G'_{p,q} = \{1\}$ and so $M = [G_{p,q} : G'_{p,q}] = (2 \cdot 2^1)/1 = 4$. For $p+q \geq 2$, $G'_{p,q} = \{1, -1\}$, so $M = [G_{p,q} : G'_{p,q}] = (2 \cdot 2^{p+q})/2 = 2^{p+q}$, as desired. ∎

Note that Maschke's Theorem 1 gives the decomposition $\mathbb{C}[G_{p,q}] = \oplus_i{}^N m_i V^{(i)}$ and from Proposition 4, one gets $|\mathbb{C}[G_{p,q}]| = \sum_{i=1}^{N} m_i{}^2$. From the above theorem, provided that M is the number of degree one representations of the group, the dimension of the group algebra $\mathbb{C}[G_{p,q}]$ can be rewritten as

$$|\mathbb{C}[G_{p,q}]| = M + \sum_{i=M+1}^{N} m_i^2. \tag{18}$$

Thus, the difference $N - M$ is the number of inequivalent irreducible representations of $G_{p,q}$ with degree two or more. This can be formally stated as the following result.

Theorem 11. *Let L be the number of inequivalent irreducible representations with degree two or more of $G_{p,q}$. (i) Let $p+q \geq 2$. If $p+q$ is even, then $L = 1$ otherwise $L = 2$. (ii) When $p+q = 1$, then $L = 0$.*

Proof. The proof follows immediately from Theorems 9 and 10. ∎

In the remainder of this section, we give the order structure and conjugacy classes of Salingaros' vee groups of orders 4, 8, and 16.

Example 6. *Consider the abelian groups $G_{1,0}$ and $G_{0,1}$. The number of conjugacy classes is $N = 2 + 2^1 = 4$ as predicted by Theorem 9, and the conjugacy classes are:*

$$K_1 = \{1\}, \quad K_2 = \{-1\}, \quad K_3 = \{e_1\}, \quad K_4 = \{-e_1\}. \tag{19}$$

Since the groups $G_{1,0}$ and $G_{0,1}$ have the same conjugacy classes, what distinguishes them is their order structure. The order structure of these groups is summarized in Table 2 where C.O.S. and G.O.S. give the center order structure and the group order structure, respectively, of each group. Also, ${}^2\mathrm{Mat}(1,\mathbb{R})$ *denotes* $\mathrm{Mat}(1,\mathbb{R}) \oplus \mathrm{Mat}(1,\mathbb{R})$.

Tab. 2: Vee groups $G_{p,q}$ of order 4 for $p+q=1$

(p,q)	Group	$C\ell_{p,q}$	Center	β^2	C.O.S.	G.O.S.	L	M	N
$(1,0)$	$G_{1,0}=D_4$	${}^2\mathrm{Mat}(1,\mathbb{R})$	$\mathbb{Z}_2\times\mathbb{Z}_2$	$+1$	$(1,3,0)$	$(1,3,0)$	0	4	4
$(0,1)$	$G_{0,1}=\mathbb{Z}_4$	$\mathrm{Mat}(1,\mathbb{C})$	$\mathbb{Z}_4$	-1	$(1,1,2)$	$(1,1,2)$	0	4	4

Example 7. *Consider the non-abelian groups $G_{2,0}$, $G_{1,1}$, and $G_{0,2}$. It is easy to check that the conjugacy classes for these groups are:*

(20) $$K_1=\{1\},\ K_2=\{-1\},\ K_3=\{e_1,-e_1\},\ K_4=\{e_2,-e_2\},\ K_5=\{e_{12},-e_{12}\}$$

which matches the formula $N=1+2^2=5$. The order structure of these groups is summarized in Table 3.

Example 8. *Consider the non-abelian groups $G_{3,0}$, $G_{2,1}$, $G_{1,2}$, and $G_{0,3}$. The number of conjugacy classes is $N=2+2^3=10$ as predicted earlier by Theorem 9. Thus, the conjugacy classes for each of these groups are:*

$$\begin{gathered}K_1=\{1\},\ K_2=\{-1\},\ K_3=\{e_{123}\},\ K_4=\{-e_{123}\},\ K_5=\{e_1,-e_1\},\\ K_6=\{e_2,-e_2\},\ K_7=\{e_3,-e_3\},\ K_8=\{e_{12},-e_{12}\},\\ K_9=\{e_{13},-e_{13}\},\ K_{10}=\{e_{23},-e_{23}\}.\end{gathered} \tag{21}$$

The order structure for each group is given in Table 4.

In the next section, we will present all irreducible representations of all distinct classes of Salingaros' vee groups of orders 4, 8, and 16.

Tab. 3: Vee groups $G_{p,q}$ of order 8 for $p+q=2$

(p,q)	Group	Class	$C\ell_{p,q}$	Center	β^2	C.O.S.	G.O.S.	L	M	N
$(2,0)$	$G_{2,0}=D_8$	N_1	$\mathrm{Mat}(2,\mathbb{R})$	$\mathbb{Z}_2$	-1	$(1,1,0)$	$(1,5,2)$	1	4	5
$(1,1)$	$G_{1,1}=D_8$	N_1	$\mathrm{Mat}(2,\mathbb{R})$	$\mathbb{Z}_2$	$+1$	$(1,1,0)$	$(1,5,2)$	1	4	5
$(0,2)$	$G_{0,2}=Q_8$	N_2	$\mathrm{Mat}(1,\mathbb{H})$	$\mathbb{Z}_2$	-1	$(1,1,0)$	$(1,1,6)$	1	4	5

3. Irreducible representations of vee groups of order 4, 8, and 16

In this section we compute all irreducible representations and their characters of Salingaros vee groups of orders 4, 8 and 16.

It is convenient and relatively easy to find decompositions and G-invariant submodules in the group algebras of the symmetric group and of its subgroups by using `SymGroupAlgebra`, a package for Maple [1]. Therefore, in order to decompose the

Tab. 4: Vee groups $G_{p,q}$ of order 16 for $p+q=3$

(p,q)	Group	Class	$C\ell_{p,q}$	Center	β^2	C.O.S.	G.O.S.	L	M	N
$(3,0)$	$G_{3,0}$	S_1	$\mathrm{Mat}(2,\mathbb{C})$	$\mathbb{Z}_4$	-1	$(1,1,2)$	$(1,7,8)$	2	8	10
$(2,1)$	$G_{2,1}$	Ω_1	${}^2\mathrm{Mat}(2,\mathbb{R})$	$\mathbb{Z}_2\times\mathbb{Z}_2$	$+1$	$(1,3,0)$	$(1,11,4)$	2	8	10
$(1,2)$	$G_{1,2}$	S_1	$\mathrm{Mat}(2,\mathbb{C})$	$\mathbb{Z}_4$	-1	$(1,1,2)$	$(1,7,8)$	2	8	10
$(0,3)$	$G_{0,3}$	Ω_2	${}^2\mathrm{Mat}(1,\mathbb{H})$	$\mathbb{Z}_2\times\mathbb{Z}_2$	$+1$	$(1,3,0)$	$(1,3,12)$	2	8	10

group algebra $\mathbb{C}[G_{p,q}]$ into irreducible submodules, for each vee group of interest, one first finds a subgroup S of the symmetric group S_n, $n=2^{1+p+q}$, isomorphic to $G_{p,q}$. For each vee group, its isomorphic copy S can be found from the Cayley's multiplication table of $G_{p,q}$. In fact, it is enough to find images of the generators of $G_{p,q}$ under the group isomorphism $G_{p,q} \overset{F}{\cong} S < S_n$. This isomorphism extends uniquely to a linear isomorphism, also denoted by F, of the corresponding group algebras $\mathbb{C}[G_{p,q}] \overset{F}{\cong} S < \mathbb{C}[S_n]$. Thus, all computations have been performed in $\mathbb{C}[S_n]$ and the results have been brought back to $\mathbb{C}[G_{p,q}]$ by F^{-1}.

In the following, we consider the irreducible representations and their characters of $G_{p,q}$ for $p+q=1,2,3$. While the character tables of all groups of order less than 32 can be found in the literature, for example in [12], the path followed here is to, first, explicitly decompose the group algebras $\mathbb{C}[G_{p,q}]$ into irreducible $G_{p,q}$-submodules by finding bases for these submodules, and, second, compute the irreducible representations for $G_{p,q}$ and their character tables using the elements of the representation theory presented earlier. In the process, one discovers a useful application for the Groebner basis technique [8] when searching for the bases in the $G_{p,q}$-invariant submodules.

3.1. Groups of order 4

Since the vee groups $G_{1,0}$ and $G_{0,1}$ of order 4 are isomorphic to $D_4 \cong \mathbb{Z}_2\times\mathbb{Z}_2$ and $\mathbb{Z}_4$, respectively, their character tables are easy to compute by hand and are well-known. For completeness and in preparation for handling groups of higher orders, it is worth to describe an algorithm for finding irreducible representations, and their characters, of these groups. Since both groups are abelian, their conjugacy classes are one-element classes.

3.1.1. The group $G_{1,0}=D_4$

The group $G_{1,0}$ is generated by -1 and e_1 with $e_1^2=1$ while the group $S\subset S_4$ isomorphic to it is generated by the permutations $(1,2)(3,4)$ and $(1,3)(2,4)$ (see Table 5). That is, $-1\mapsto(1,2)(3,4)$ and $e_1\mapsto(1,3)(2,4)$ under the isomorphism F mentioned above. For completeness, $-e_1\mapsto(1,4)(2,3)$ and $1\mapsto(1)$.

One needs to find four vectors $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$, and $\mathbf{u}_4$ which span 1-dimensional $G_{1,0}$-invariant subspaces $V^{(1)}$, $V^{(2)}$, $V^{(3)}$, and $V^{(4)}$ such that

$$\mathbb{C}[G_{1,0}] = V^{(1)} \oplus V^{(2)} \oplus V^{(3)} \oplus V^{(4)} \tag{22}$$

and $V^{(i)} = \operatorname{span}\{\mathbf{u}_i\}$, $i = 1, \ldots, 4$. Notice that all $V^{(i)}$ are of dimension 1 since the group is abelian and all irreducible modules are one dimensional. The following algorithm can be used to find the basis vectors.

Algorithm 1.

1: Let $G = S \cong G_{1,0}$ and $V = \mathbb{C}[S] \cong \mathbb{C}[G_{1,0}]$.

2: Let $\mathbf{u}_1$ be the sum of all basis elements in V and define $V^{(1)} = \operatorname{span}\{\mathbf{u}_1\}$. Such subspace always carries the trivial representation and it is G-invariant since $g\mathbf{u}_1 = \mathbf{u}_1$ for every $g \in G$.

3: Compute a basis for the orthogonal complement of $V^{(1)}$ in V and rename this complement as V. This orthogonal complement is obviously 3-dimensional and it is G-invariant by Proposition 1.

4: Using Groebner basis technique [8], find a 1-dimensional G-invariant subspace $V^{(2)}$ in V and find its spanning vector $\mathbf{u}_2$.

5: Find a 2-dimensional orthogonal complement of $V^{(2)}$ in V. Call this complement V. By the same reasoning, it is G-invariant.

6: Find a 1-dimensional G-invariant subspace $V^{(3)}$ in V different from $V^{(2)}$ and its spanning vector $\mathbf{u}_3$.

7: Find a basis for the orthogonal complement $V^{(4)}$ of $V^{(1)} \oplus V^{(2)} \oplus V^{(3)}$ in $\mathbb{C}[G_{1,0}]$ and its spanning vector $\mathbf{u}_4$.

8: The algorithm terminates since the dimension of $\mathbb{C}[G_{1,0}]$ is finite.

From the above procedure, one obtains all basis vectors $\mathbf{u}_i$ as linear combinations of the standard basis $\mathcal{B} = \{1, -1, e_1, -e_1\}$ of $\mathbb{C}[G_{1,0}]$ as follows:

$$\begin{aligned}
V^{(1)} &= \operatorname{span}\{\mathbf{u}_1\}, &\quad \mathbf{u}_1 &= (1)1 + (1)(-1) + (1)(e_1) + (1)(-e_1),\\
V^{(2)} &= \operatorname{span}\{\mathbf{u}_2\}, &\quad \mathbf{u}_2 &= (1)1 + (-1)(-1) + (-1)(e_1) + (1)(-e_1),\\
V^{(3)} &= \operatorname{span}\{\mathbf{u}_3\}, &\quad \mathbf{u}_3 &= (-1)1 + (1)(-1) + (-1)(e_1) + (1)(-e_1),\\
V^{(4)} &= \operatorname{span}\{\mathbf{u}_4\}, &\quad \mathbf{u}_4 &= (-1)1 + (-1)(-1) + (1)(e_1) + (1)(-e_1).
\end{aligned} \tag{23}$$

Once the decomposition (22) has been determined, one can find all four irreducible inequivalent representations $X^{(1)}$, $X^{(2)}$, $X^{(3)}$, and $X^{(4)}$ in the corresponding subspaces $V^{(1)}$, $V^{(2)}$, $V^{(3)}$, and $V^{(4)}$. These are all 1-dimensional and can be read off from the following character table.

(24)

char/class	K_1	K_2	K_3	K_4
$\chi^{(1)}$	1	1	1	1
$\chi^{(2)}$	1	-1	-1	1
$\chi^{(3)}$	1	-1	1	-1
$\chi^{(4)}$	1	1	-1	-1

The explicit matrix representations are shown in Table 12 in Appendix B. Note that in the character table, rows and columns are orthonormal. Let $\chi^{(i)}$ denote the character of the representation $X^{(i)}$. So, for example, the inner product of the characters $\chi^{(2)}$ and $\chi^{(3)}$ from the above table is computed as follows:

$$\langle \chi^{(2)}, \chi^{(3)} \rangle = \frac{1}{4} \sum_{i=1}^{4} |K_i| \chi^{(2)}_{K_i} \overline{\chi^{(3)}_{K_i}} = \frac{1}{4}((1)(1) + (-1)(-1) + (-1)(1) + (1)(-1)) = 0$$

since $|K_i| = 1$ for each class. This verifies the character orthogonality relation of the first kind. In a similar manner one can verify the character relation of the second kind.

3.1.2. The group $G_{0,1} = \mathbb{Z}_4$

The group $G_{0,1}$ is generated by -1 and e_1 with $e_1^2 = -1$ while the group $S \subset S_4$ isomorphic to it is generated by the permutations $(1,2)(3,4)$ and $(1,3,2,4)$ (see Table 5). That is, $-1 \mapsto (1,2)(3,4)$ and $e_1 \mapsto (1,3,2,4)$ under the isomorphism F. For completeness, $-e_1 \mapsto (1,4,2,3)$ and $1 \mapsto (1)$.

One needs to again find four vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4$ which span 1-dimensional $G_{0,1}$-invariant subspaces $V^{(1)}$, $V^{(2)}$, $V^{(3)}$, and $V^{(4)}$ such that

$$\mathbb{C}[G_{0,1}] = V^{(1)} \oplus V^{(2)} \oplus V^{(3)} \oplus V^{(4)} \tag{25}$$

and $V^{(i)} = \mathrm{span}\{\mathbf{u}_i\}$, $i = 1, \ldots, 4$. The subspaces $V^{(i)}$ again are of dimension 1 since $G_{0,1}$ is abelian and so all its irreducible modules are one dimensional. Applying now Algorithm 1 to $\mathbb{C}[G_{0,1}]$, one finds this basis:

$$\begin{aligned}
V^{(1)} &= \mathrm{span}\{\mathbf{u}_1\}, & \mathbf{u}_1 &= (1)(1) + (1)(-1) + (1)(e_1) + (1)(-e_1),\\
V^{(2)} &= \mathrm{span}\{\mathbf{u}_2\}, & \mathbf{u}_2 &= (-i)1 + (i)(-1) + (-1)(e_1) + (1)(-e_1),\\
V^{(3)} &= \mathrm{span}\{\mathbf{u}_3\}, & \mathbf{u}_3 &= (-1)1 + (-1)(-1) + (1)(e_1) + (-1)(-e_1),\\
V^{(4)} &= \mathrm{span}\{\mathbf{u}_4\}, & \mathbf{u}_4 &= (i)1 + (-i)(-1) + (-1)(e_1) + (1)(-e_1).
\end{aligned} \tag{26}$$

Once the decomposition (25) has been found, one can determine all four irreducible inequivalent representations $X^{(1)}$, $X^{(2)}$, $X^{(3)}$, and $X^{(4)}$ in the corresponding subspaces $V^{(1)}$, $V^{(2)}$, $V^{(3)}$, and $V^{(4)}$. These are all 1-dimensional and can be read off from the character table.

(27)

char/class	K_1	K_2	K_3	K_4
$\chi^{(1)}$	1	1	1	1
$\chi^{(2)}$	1	-1	i	$-i$
$\chi^{(3)}$	1	1	-1	-1
$\chi^{(4)}$	1	-1	$-i$	i

The explicit matrix representations are shown in Table 13 in Appendix B. Like in the previous example, one can verify that the columns and rows in the above character table are orthonormal. This is in agreement with the character orthogonality relations of the first and of the second kind.

3.2. Groups of order 8

Since the vee groups $G_{2,0}$, $G_{1,1}$ and $G_{0,2}$ are isomorphic to D_8, D_8 and Q_8, respectively, their character tables are easy to compute by hand and are well-known. However, computation of their representations is not so simple. For completeness and in preparation for handling groups of order 16, we describe an algorithm for finding irreducible representations and characters of these groups. Note that the conjugacy classes for these groups are shown in (20).

3.2.1. The extra-special group $G_{2,0} = D_8 = N_1$

The group $G_{2,0}$ is generated by -1, e_1 and e_2 with $e_1^2 = e_2^2 = 1$, $e_1 e_2 = -e_2 e_1$, while the group $S \subset S_8$ isomorphic to $G_{2,0}$ is generated by the permutations $(1,2)(3,4)(5,6)(7,8)$, $(1,3)(2,4)(5,7)(6,8)$ and $(1,5)(2,6)(3,8)(4,7)$ (see Table 6). That is,

$$-1 \mapsto (1,2)(3,4)(5,6)(7,8),\ e_1 \mapsto (1,3)(2,4)(5,7)(6,8),\ e_2 \mapsto (1,5)(2,6)(3,8)(4,7)$$

under the isomorphism F mentioned above.

In a manner similar to $G_{1,0}$, we describe an algorithm for finding the decomposition of $\mathbb{C}[G_{2,0}]$ into invariant subspaces

$$\mathbb{C}[G_{2,0}] = \bigoplus_{i=1}^{6} V^{(i)} \tag{28}$$

where $V^{(i)} = \operatorname{span}\{\mathbf{u}_i\}$, $i = 1, \ldots, 4$, are one-dimensional subspaces while $V^{(5)}$ and $V^{(6)}$ are two-dimensional subspaces carrying equivalent representations according to Proposition 4, Theorem 10 and Theorem 11.

The basis vectors $\mathbf{u}_i, i = 1, \ldots, 8$, are displayed in (43) in Appendix B. They have been found by using the following algorithm.

Algorithm 2.

1: Let $G = S \cong G_{2,0}$ and $V = \mathbb{C}[S] \cong \mathbb{C}[G_{2,0}]$.

2: Apply Algorithm 1 to find vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4$ *providing bases for the one-dimensional G-invariant submodules* $V^{(1)}, V^{(2)}, V^{(3)}, V^{(4)}$ *in* V.

3: Find a basis for the orthogonal complement of $V^{(1)} \oplus V^{(2)} \oplus V^{(3)} \oplus V^{(4)}$ *in* V *and call it* V. *It is* 4*-dimensional.*

4: Using Groebner basis technique, find any 2*-dimensional* G*-invariant subspace in* V *and call it* $V^{(5)}$. *That is, find its basis vectors* $\mathbf{u}_5$ *and* $\mathbf{u}_6$.

5: Find a basis for the orthogonal complement of $V^{(5)}$ *in* V *and call it* $V^{(6)}$. *That is, find its spanning vectors* $\mathbf{u}_7$ *and* $\mathbf{u}_8$.

6: The algorithm terminates when all eight vectors $\mathbf{u}_1, \ldots, \mathbf{u}_8$ *are found and these vectors provide a basis for the decomposition of* $\mathbb{C}[G_{2,0}]$.

Once the decomposition of $\mathbb{C}[G_{2,0}]$ has been found, one can compute all irreducible representations $X^{(i)}, i = 1, \ldots, 6$, of $G_{2,0}$ in the six invariant submodules $V^{(i)}$. The degree-one representations $X^{(1)}$, $X^{(2)}$, $X^{(3)}$, and $X^{(4)}$ are all inequivalent since their characters are different as shown in the character table below. The two irreducible representations $X^{(5)}$ and $X^{(6)}$ of degree two are equivalent. All representations are displayed in Table 14 in Appendix B. The extended character table with all representations, including the equivalent ones, for $G_{2,0}$ is as follows:

(29)

char/class	K_1	K_2	K_3	K_4	K_5
$\chi^{(1)}$	1	1	1	1	1
$\chi^{(2)}$	1	1	-1	-1	1
$\chi^{(3)}$	1	1	1	-1	-1
$\chi^{(4)}$	1	1	-1	1	-1
$\chi^{(5)}$	2	-2	0	0	0
$\chi^{(6)}$	2	-2	0	0	0

Note that $X^{(5)}$ and $X^{(6)}$ are equivalent since their characters are the same. To illustrate the character orthogonality relations, compute the inner product of the characters $\chi^{(2)}$ and $\chi^{(3)}$ while taking into consideration the number of elements in each conjugacy class:

$$\begin{aligned}
\langle \chi^{(2)}, \chi^{(3)} \rangle &= \frac{1}{8} \sum_{i=1}^{5} |K_i| \chi^{(2)}_{K_i} \overline{\chi^{(3)}_{K_i}} \\
&= \frac{1}{8}(1 \cdot (1)(1) + 1 \cdot (1)(1) + 2 \cdot (-1)(1) + 2 \cdot (-1)(-1) + 2 \cdot (1)(-1)) \\
&= 0. \qquad (30)
\end{aligned}$$

In a similar manner one can verify the character relation of the second kind.

Since the group $G_{1,1}$ since it belongs to the same class N_1 as $G_{2,0}$, it will be not discussed separately.

3.2.2. The extra-special group $G_{0,2} = Q_8 = N_2$

The group $G_{0,2}$ is generated by -1, e_1 and e_2 with $e_1^2 = e_2^2 = -1$, $e_1 e_2 = -e_2 e_1$ while the group $S \subset S_8$ isomorphic to $G_{0,2}$ is generated by the permutations $(1,2)(3,4)(5,6)(7,8)$, $(1,3,2,4)(5,7,6,8)$ and $(1,5,2,6)(3,8,4,7)$ (see Table 6). So,

$$-1 \mapsto (1,2)(3,4)(5,6)(7,8),\ e_1 \mapsto (1,3,2,4)(5,7,6,8),\ e_2 \mapsto (1,5,2,6)(3,8,4,7)$$

under the isomorphism F.

Since the group algebra $\mathbb{C}[G_{0,2}]$ formally decomposes like (28), using Algorithm 2 one can find the decomposition of $\mathbb{C}[G_{0,2}]$. The basis vectors spanning the invariant subspaces are shown in (45) in Appendix B. Once the decomposition has been found, one can compute all six irreducible representations $X^{(i)}$, $i = 1, \ldots 6$, in the corresponding submodules $V^{(i)}$. These representations are shown in Table 15 in Appendix B. The character table with all representations for $G_{0,2}$ is as follows:

(31)

char/class	K_1	K_2	K_3	K_4	K_5
$\chi^{(1)}$	1	1	1	1	1
$\chi^{(2)}$	1	1	-1	1	-1
$\chi^{(3)}$	1	1	1	-1	-1
$\chi^{(4)}$	1	1	-1	-1	1
$\chi^{(5)}$	2	-2	0	0	0
$\chi^{(6)}$	2	-2	0	0	0

Note that since the characters of $X^{(5)}$ and $X^{(6)}$ are the same, these representations are equivalent. The character table without the last row again shows that the characters satisfy the two orthogonality relations.

3.3. Groups of order 16

In this section we discuss the vee groups $G_{3,0}$, $G_{2,1}$, $G_{1,2}$ and $G_{0,3}$. Conjugacy classes of these groups are given in (21). Their character tables can be computed using a combination of Algorithm 1 and Algorithm 2, as required.

3.3.1. The group $G_{3,0} = S_1$

The group $G_{3,0}$ is generated by -1, e_1, e_2 and e_3 with $e_1^2 = e_2^2 = e_3^2 = 1$, $e_i e_j = -e_j e_i$, $i \neq j$, while the group $S \subset S_{16}$ isomorphic to $G_{3,0}$ and generated by corresponding permutations is shown in Table 8 in Appendix A.

In a manner similar to the groups of orders 4 and 8, one can find the decomposition of $\mathbb{C}[G_{3,0}]$ into invariant subspaces

$$\mathbb{C}[G_{3,0}] = \bigoplus_{i=1}^{12} V^{(i)} \tag{32}$$

where $V^{(i)} = \mathrm{span}\{\mathbf{u}_i\}$, $i = 1, \ldots, 8$, are one-dimensional while

$$\begin{aligned} V^{(9)} &= \mathrm{span}\{\mathbf{u}_9, \mathbf{u}_{10}\}, \quad V^{(10)} = \mathrm{span}\{\mathbf{u}_{11}, \mathbf{u}_{12}\}, \\ V^{(11)} &= \mathrm{span}\{\mathbf{u}_{13}, \mathbf{u}_{14}\}, \quad V^{(12)} = \mathrm{span}\{\mathbf{u}_{15}, \mathbf{u}_{16}\} \end{aligned} \tag{33}$$

are two-dimensional subspaces carrying two pairwise equivalent representations according to Proposition 4, Theorem 10 and Theorem 11.

The basis vectors $\mathbf{u}_i$ are displayed in (47) in Appendix B. They have been found by using the above two algorithms.

Once the decomposition of $\mathbb{C}[G_{3,0}]$ has been determined, one can compute all irreducible representations $X^{(i)}$ of $G_{3,0}$. The representations are displayed in Table 16 in Appendix B. The extended character table for $G_{3,0}$ is as follows:

(34)

char/class	K_1	K_2	K_3	K_4	K_5	K_6	K_7	K_8	K_9	K_{10}
$\chi^{(1)}$	1	1	1	1	1	1	1	1	1	1
$\chi^{(2)}$	1	1	-1	-1	1	1	-1	1	-1	-1
$\chi^{(3)}$	1	1	-1	-1	1	-1	1	-1	1	-1
$\chi^{(4)}$	1	1	1	1	1	-1	-1	-1	-1	1
$\chi^{(5)}$	1	1	-1	-1	-1	1	1	-1	-1	1
$\chi^{(6)}$	1	1	1	1	-1	1	-1	-1	1	-1
$\chi^{(7)}$	1	1	1	1	-1	-1	1	1	-1	-1
$\chi^{(8)}$	1	1	-1	-1	-1	-1	-1	1	1	1
$\chi^{(9)}$	2	-2	$2i$	$-2i$	0	0	0	0	0	0
$\chi^{(10)}$	2	-2	$-2i$	$2i$	0	0	0	0	0	0
$\chi^{(11)}$	2	-2	$-2i$	$2i$	0	0	0	0	0	0
$\chi^{(12)}$	2	-2	$2i$	$-2i$	0	0	0	0	0	0

Note that $X^{(9)} \cong X^{(12)}$ and $X^{(9)} \cong X^{(12)}$ since their characters are the same. To illustrate orthogonality of the characters, consider the inner product of the characters $\chi^{(2)}$ and $\chi^{(3)}$:

$$\begin{aligned} \langle \chi^{(2)}, \chi^{(3)} \rangle &= \frac{1}{16} \sum_{i=1}^{10} |K_i| \chi^{(2)}_{K_i} \overline{\chi^{(3)}_{K_i}} \\ &= \frac{1}{16}(1 \cdot (1)(1) + 1 \cdot (1)(1) + 1 \cdot (-1)(-1) + 1 \cdot (-1)(-1) \\ &\quad + 2 \cdot (1)(1) + 2 \cdot (1)(-1) + 2 \cdot (-1)(1) \\ &\quad + 2 \cdot (1)(-1) + 2 \cdot (-1)(1) + 2 \cdot (-1)(-1)) \\ &= 0. \end{aligned} \tag{35}$$

which verifies the character orthogonality relation of the first kind. In a similar manner one can verify the character relation of the second kind.

Since the group $G_{1,2}$ belongs to the same class S_1 as $G_{3,0}$, it will not be discussed separately.

3.3.2. The group $G_{2,1} = \Omega_1$

The group $G_{2,1}$ is generated by -1, e_1, e_2 and e_3 with $e_1^2 = e_2^2 = 1$ and $e_3^2 = -1$, $e_i e_j = -e_j e_i$, $i \neq j$, while the group $S \subset S_{16}$ isomorphic to $G_{2,1}$ is generated by the permutations of S_{16} as shown in Table 9 in Appendix A.

The decomposition of $\mathbb{C}[G_{2,1}]$ looks the same as that of $\mathbb{C}[G_{3,0}]$ displayed in (32), while the basis vectors $\mathbf{u}_i$ for this decomposition are displayed in (49) in Appendix B. They have been found by using the above two algorithms.

Once the decomposition of $\mathbb{C}[G_{2,1}]$ has been found, one can compute all irreducible representations $X^{(i)}$ of $G_{2,1}$. The representations are displayed in Table 17 in Appendix B. The extended character table for $G_{2,1}$ is as follows:

(36)

char/class	K_1	K_2	K_3	K_4	K_5	K_6	K_7	K_8	K_9	K_{10}
$\chi^{(1)}$	1	1	1	1	1	1	1	1	1	1
$\chi^{(2)}$	1	1	-1	-1	1	1	-1	1	-1	-1
$\chi^{(3)}$	1	1	-1	-1	1	-1	1	-1	1	-1
$\chi^{(4)}$	1	1	1	1	1	-1	-1	-1	-1	1
$\chi^{(5)}$	1	1	-1	-1	-1	1	1	-1	-1	1
$\chi^{(6)}$	1	1	1	1	-1	1	-1	-1	1	-1
$\chi^{(7)}$	1	1	1	1	-1	-1	1	1	-1	-1
$\chi^{(8)}$	1	1	-1	-1	-1	-1	-1	1	1	1
$\chi^{(9)}$	2	-2	-2	2	0	0	0	0	0	0
$\chi^{(10)}$	2	-2	2	-2	0	0	0	0	0	0
$\chi^{(11)}$	2	-2	-2	2	0	0	0	0	0	0
$\chi^{(12)}$	2	-2	2	-2	0	0	0	0	0	0

Note that $X^{(9)} \cong X^{(11)}$ and $X^{(10)} \cong X^{(12)}$ since their characters are the same.

3.3.3. The group $G_{0,3} = \Omega_2$

The group $G_{0,3}$ is generated by -1, e_1, e_2 and e_3 with $e_1^2 = e_2^2 = e_3^2 = -1$, $e_i e_j = -e_j e_i, i \neq j$, while the group $S \subset S_{16}$ isomorphic to $G_{0,3}$ is generated by the corresponding permutations of S_{16} shown in Table 11 in Appendix A.

The decomposition of $\mathbb{C}[G_{0,3}]$ again looks the same as that of $\mathbb{C}[G_{3,0}]$ in (32), while the basis vectors $\mathbf{u}_i$ are displayed in (51) in Appendix B. They have been found by using the above two algorithms. The irreducible representations $X^{(i)}$ of $G_{0,3}$ have been computed in the same manner. They are displayed in Table 17 in Appendix B.

The extended character table for $G_{0,3}$ is as follows:

(37)

char/class	K_1	K_2	K_3	K_4	K_5	K_6	K_7	K_8	K_9	K_{10}
$\chi^{(1)}$	1	1	1	1	1	1	1	1	1	1
$\chi^{(2)}$	1	1	−1	−1	1	1	−1	1	−1	−1
$\chi^{(3)}$	1	1	−1	−1	1	−1	1	−1	1	−1
$\chi^{(4)}$	1	1	1	1	1	−1	−1	−1	−1	1
$\chi^{(5)}$	1	1	−1	−1	−1	1	1	−1	−1	1
$\chi^{(6)}$	1	1	1	1	−1	1	−1	−1	1	−1
$\chi^{(7)}$	1	1	1	1	−1	−1	1	1	−1	−1
$\chi^{(8)}$	1	1	−1	−1	−1	−1	−1	1	1	1
$\chi^{(9)}$	2	−2	−2	2	0	0	0	0	0	0
$\chi^{(10)}$	2	−2	2	−2	0	0	0	0	0	0
$\chi^{(11)}$	2	−2	−2	2	0	0	0	0	0	0
$\chi^{(12)}$	2	−2	2	−2	0	0	0	0	0	0

Note that $X^{(9)} \cong X^{(11)}$ and $X^{(10)} \cong X^{(12)}$ since their characters are the same.

4. Conclusions

Due to the renewed interest in the relationship between finite Salingaros' vee groups $G = G_{p,q}$ and Clifford algebras, the main goal of this paper has been to show how one can construct irreducible representations of these groups by decomposing their regular modules. In the process, two algorithms have been formulated which have allowed us to completely decompose regular modules of groups of orders 4, 8, and 16 into irreducible G-submodules. These algorithms have used Groebner basis approach to find bases in these G-submodules as well as the G-invariance of an inner product defined on the complex regular module $\mathbb{C}G$. In the process, we have computed character tables of these groups. Of course, the character tables of these groups are known and can be found in the literature, e.g., see [12] and references therein. It is much more efficient to derive the character tables using the character theory instead of finding the actual representations first. Furthermore, knowing the irreducible characters of a finite group G, one can use them to decompose any G-module, let it be regular or, for example, a permutation module, into a direct sum of G-submodules without a common composition factor. This approach is based on defining, for each irreducible character χ_i of G an idempotent element e_i in the group algebra $\mathbb{C}G$ such that these (not necessarily primitive) idempotents provide an orthogonal decomposition of the unity in $\mathbb{C}G$. This way, for example, the regular module $\mathbb{C}G$ can be decomposed into a direct sum of two sided ideals $\mathbb{C}Ge_i$ generated by the idempotents. These ideals as G-submodules, do not share a common composition factor and are reducible if the degree of χ_i is greater than 1. [12] Then, to achieve a complete decomposition, these reducible G-modules can be further decomposed by an algorithm similar to the Algorithm 2. The two algorithms can be applied to groups of higher order than 16, if needed.

A. Images of the generators of the vee groups

In this Appendix, we show images of the generators of the vee groups $G_{p,q}$ for $p+q \leq 3$ in the symmetric groups S_n where $n = 2^{1+p+q}$.

Tab. 5: Generators for $G_{1,0}$ and $G_{0,1}$ in S_4

	$G_{1,0}$	Order	$G_{0,1}$	Order
-1	$(1,2)(3,4)$	2	$(1,2)(3,4)$	2
e_1	$(1,3)(2,4)$	2	$(1,3,2,4)$	4

Tab. 6: Generators for $G_{2,0}$ and $G_{1,1}$ in S_8

	$G_{2,0}$	Order	$G_{1,1}$	Order
-1	$(1,2)(3,4)(5,6)(7,8)$	2	$(1,2)(3,4)(5,6)(7,8)$	2
e_1	$(1,3)(2,4)(5,7)(6,8)$	2	$(1,3)(2,4)(5,7)(6,8)$	2
e_2	$(1,5)(2,6)(3,8)(4,7)$	2	$(1,5,2,6)(3,8,4,7)$	4

Tab. 7: Generators for $G_{0,2}$ in S_8

	$G_{0,2}$	Order
-1	$(1,2)(3,4)(5,6)(7,8)$	2
e_1	$(1,3,2,4)(5,7,6,8)$	4
e_2	$(1,5,2,6)(3,8,4,7)$	4

Tab. 8: Generators for $G_{3,0}$ in S_{16}

	$G_{3,0}$	Order
-1	$(1,2)(3,4)(5,6)(7,8)(9,10)(11,12)(13,14)(15,16)$	2
e_1	$(1,3)(2,4)(5,9)(6,10)(7,11)(8,12)(13,15)(14,16)$	2
e_2	$(1,5)(2,6)(3,10)(4,9)(7,13)(8,14)(11,16)(12,15)$	2
e_3	$(1,7)(2,8)(3,12)(4,11)(5,14)(6,13)(9,15)(10,16)$	2

B. Irreducible representations of the vee groups

In this Appendix, we show matrices for one representative g from each conjugacy class K_i in each irreducible representation $X^{(j)}$ for all vee groups of orders 4, 8, and 16. Here, the index i runs through all conjugacy classes whereas the index j runs through all irreducible representations including equivalent ones.

For consistency, matrices shown in the tables below always represent the first element in each class.

Tab. 9: Generators for $G_{2,1}$ in S_{16}

	$G_{2,1}$	Order
-1	$(1,2)(3,4)(5,6)(7,8)(9,10)(11,12)(13,14)(15,16)$	2
e_1	$(1,3)(2,4)(5,9)(6,10)(7,11)(8,12)(13,15)(14,16)$	2
e_2	$(1,5)(2,6)(3,10)(4,9)(7,13)(8,14)(11,16)(12,15)$	2
e_3	$(1,7,2,8)(3,12,4,11)(5,14,6,13)(9,15,10,16)$	4

Tab. 10: Generators for $G_{1,2}$ in S_{16}

	$G_{1,2}$	Order
-1	$(1,2)(3,4)(5,6)(7,8)(9,10)(11,12)(13,14)(15,16)$	2
e_1	$(1,3)(2,4)(5,9)(6,10)(7,11)(8,12)(13,15)(14,16)$	2
e_2	$(1,5,2,6)(3,10,4,9)(7,13,8,14)(11,16,12,15)$	4
e_3	$(1,7,2,8)(3,12,4,11)(5,14,6,13)(9,15,10,16)$	4

Tab. 11: Generators for $G_{0,3}$ in S_{16}

	$G_{0,3}$	Order
-1	$(1,2)(3,4)(5,6)(7,8)(9,10)(11,12)(13,14)(15,16)$	2
e_1	$(1,3,2,4)(5,9,6,10)(7,11,8,12)(13,15,14,16)$	4
e_2	$(1,5,2,6)(3,10,4,9)(7,13,8,14)(11,16,12,15)$	4
e_3	$(1,7,2,8)(3,12,4,11)(5,14,6,13)(9,15,10,16)$	4

For the groups of order 4, all representations are inequivalent, and are shown in Tables 12 and 13.

In Table 12, the irreducible representations $X^{(i)}$ of $G_{1,0}$ are realized in irreducible $G_{1,0}$-invariant submodules of the group algebra $\mathbb{C}[G_{1,0}]$ which is decomposed as follows:

$$\mathbb{C}[G_{1,0}] = V^{(1)} \oplus V^{(2)} \oplus V^{(3)} \oplus V^{(4)}. \tag{38}$$

The one-dimensional submodules $V^{(i)}$ are spanned by the corresponding vectors $\mathbf{u}_i$, $i = 1, \ldots, 4$. The coordinates of these vectors in the basis $\mathcal{B} = \{1, -1, e_1, -e_1\}$ are as follows (:

$$\begin{aligned} V^{(1)} &= \operatorname{span}\{\mathbf{u}_1\}, \qquad \mathbf{u}_1 = (1,1,1,1),\\ V^{(2)} &= \operatorname{span}\{\mathbf{u}_2\}, \qquad \mathbf{u}_2 = (1,-1,-1,1),\\ V^{(3)} &= \operatorname{span}\{\mathbf{u}_3\}, \qquad \mathbf{u}_3 = (-1,1,-1,1),\\ V^{(4)} &= \operatorname{span}\{\mathbf{u}_4\}, \qquad \mathbf{u}_4 = (-1,-1,1,1). \end{aligned} \tag{39}$$

In Table 13, the irreducible representations $X^{(i)}$ of $G_{0,1}$ are realized in irreducible $G_{0,1}$-invariant submodules of the group algebra $\mathbb{C}[G_{0,1}]$ which is decomposed as

Tab. 12: Representations of $G_{1,0} = D_4$

	K_1	K_2	K_3	K_4
g	1	-1	e_1	$-e_1$
$X^{(1)}$	(1)	(1)	(1)	(1)
$X^{(2)}$	(1)	(-1)	(-1)	(1)
$X^{(3)}$	(1)	(-1)	(1)	(-1)
$X^{(4)}$	(1)	(1)	(-1)	(-1)

follows:

$$\mathbb{C}[G_{0,1}] = V^{(1)} \oplus V^{(2)} \oplus V^{(3)} \oplus V^{(4)}. \tag{40}$$

The one-dimensional submodules $V^{(i)}$ are spanned by the corresponding vectors $\mathbf{u}_i$, $i = 1, \dots, 4$. The complex-valued coordinates of these vectors in the basis $\mathcal{B} = \{1, -1, e_1, -e_1\}$ are as follows:

$$\begin{aligned} V^{(1)} &= \operatorname{span}\{\mathbf{u}_1\}, & \mathbf{u}_1 &= (1, 1, 1, 1), \\ V^{(2)} &= \operatorname{span}\{\mathbf{u}_2\}, & \mathbf{u}_2 &= (i, -i, -1, 1), \\ V^{(3)} &= \operatorname{span}\{\mathbf{u}_3\}, & \mathbf{u}_3 &= (-1, -1, 1, -1), \\ V^{(4)} &= \operatorname{span}\{\mathbf{u}_4\}, & \mathbf{u}_4 &= (i, -i, -1, 1). \end{aligned} \tag{41}$$

Tab. 13: Representations of $G_{0,1} = \mathbb{Z}_4$

	K_1	K_2	K_3	K_4
g	1	-1	e_1	$-e_1$
$X^{(1)}$	(1)	(1)	(1)	(1)
$X^{(2)}$	(1)	(-1)	(i)	$(-i)$
$X^{(3)}$	(1)	(1)	(-1)	(-1)
$X^{(4)}$	(1)	(-1)	$(-i)$	(i)

In Table 14, the irreducible representations $X^{(i)}$ of $G_{2,0} = D_8 = N_1$ are realized in irreducible $G_{2,0}$-invariant submodules of the group algebra $\mathbb{C}[G_{2,0}]$ which is decomposed as follows:

$$\mathbb{C}[G_{2,0}] = V^{(1)} \oplus V^{(2)} \oplus V^{(3)} \oplus V^{(4)} \oplus V^{(5)} \oplus V^{(6)}. \tag{42}$$

The submodules $V^{(i)}$ are spanned by the corresponding vectors $\mathbf{u}_i$, $i = 1, \dots, 8$, as shown below. The coordinates of these vectors in the standard basis

$$\mathcal{B} = \{1, -1, e_1, -e_1, e_2, -e_2, e_{12}, -e_{12}\}$$

are as follows:

$$
\begin{aligned}
V^{(1)} &= \operatorname{span}\{\mathbf{u}_1\}, & \mathbf{u}_1 &= (1,1,1,1,1,1,1,1),\\
V^{(2)} &= \operatorname{span}\{\mathbf{u}_2\}, & \mathbf{u}_2 &= (-1,-1,1,1,1,1,-1,-1),\\
V^{(3)} &= \operatorname{span}\{\mathbf{u}_3\}, & \mathbf{u}_3 &= (-1,-1,-1,-1,1,1,1,1),\\
V^{(4)} &= \operatorname{span}\{\mathbf{u}_4\}, & \mathbf{u}_4 &= (1,1,-1,-1,1,1,-1,-1),\\
V^{(5)} &= \operatorname{span}\{\mathbf{u}_5,\mathbf{u}_6\}, & \mathbf{u}_5 &= (-1,1,-1,1,-1,1,1,-1),\\
 & & \mathbf{u}_6 &= (-5,5,-5,5,-1,1,1,-1),\\
V^{(6)} &= \operatorname{span}\{\mathbf{u}_7,\mathbf{u}_8\}, & \mathbf{u}_7 &= (1,-1,-1,1,0,0,0,0),\\
 & & \mathbf{u}_8 &= (1,-1,-1,1,-1,1,-1,1).
\end{aligned} \tag{43}
$$

While the one-dimensional representations $X^{(1)}, X^{(2)}, X^{(3)}, X^{(4)}$ are inequivalent, the two-dimensional representations $X^{(5)}$ and $X^{(6)}$ are equivalent.

Tab. 14: Representations of $G_{2,0} = D_4 = N_1$

	K_1	K_2	K_3	K_4	K_5
g	1	-1	e_1	e_2	e_{12}
$X^{(1)}$	(1)	(1)	(1)	(1)	(1)
$X^{(2)}$	(1)	(1)	(-1)	(-1)	(1)
$X^{(3)}$	(1)	(1)	(1)	(-1)	(-1)
$X^{(4)}$	(1)	(1)	(-1)	(1)	(-1)
$X^{(5)}$	$\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$	$\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}$	$\begin{pmatrix}-\frac{3}{2} & -\frac{5}{2}\\ \frac{1}{2} & \frac{3}{2}\end{pmatrix}$	$\begin{pmatrix}1 & 6\\ 0 & -1\end{pmatrix}$	$\begin{pmatrix}-\frac{3}{2} & -\frac{13}{2}\\ \frac{1}{2} & \frac{3}{2}\end{pmatrix}$
$X^{(6)}$	$\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$	$\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}$	$\begin{pmatrix}-1 & -2\\ 0 & 1\end{pmatrix}$	$\begin{pmatrix}1 & 0\\ -1 & -1\end{pmatrix}$	$\begin{pmatrix}1 & 2\\ -1 & -1\end{pmatrix}$

In Table 15, the irreducible representations $X^{(i)}$ of $G_{0,2} = Q_8 = N_2$ are realized in irreducible $G_{0,2}$-invariant submodules of the group algebra $\mathbb{C}[G_{0,2}]$ which is decomposed as follows:

$$\mathbb{C}[G_{0,2}] = V^{(1)} \oplus V^{(2)} \oplus V^{(3)} \oplus V^{(4)} \oplus V^{(5)} \oplus V^{(6)}. \tag{44}$$

The submodules $V^{(i)}$ are spanned by the corresponding vectors $\mathbf{u}_i$, $i = 1, \ldots, 8$, as shown below. The coordinates of these vectors in the standard basis

$$\mathcal{B} = \{1, -1, e_1, -e_1, e_2, -e_2, e_{12}, -e_{12}\}$$

are as follows:

$$
\begin{aligned}
V^{(1)} &= \operatorname{span}\{\mathbf{u}_1\}, & \mathbf{u}_1 &= (1,1,1,1,1,1,1,1),\\
V^{(2)} &= \operatorname{span}\{\mathbf{u}_2\}, & \mathbf{u}_2 &= (-1,-1,1,1,-1,-1,1,1),\\
V^{(3)} &= \operatorname{span}\{\mathbf{u}_3\}, & \mathbf{u}_3 &= (-1,-1,-1,-1,1,1,1,1),\\
V^{(4)} &= \operatorname{span}\{\mathbf{u}_4\}, & \mathbf{u}_4 &= (1,1,-1,-1,-1,-1,1,1),\\
V^{(5)} &= \operatorname{span}\{\mathbf{u}_5,\mathbf{u}_6\}, & \mathbf{u}_5 &= (0,0,0,0,-i,i,1,-1),\\
& & \mathbf{u}_6 &= (-i,i,-1,1,-i,i,1,-1),\\
V^{(6)} &= \operatorname{span}\{\mathbf{u}_7,\mathbf{u}_8\}, & \mathbf{u}_7 &= (1,-1,i,-i,0,0,0,0),\\
& & \mathbf{u}_8 &= (0,0,0,0,1,-1,-i,i).
\end{aligned} \tag{45}
$$

While the one-dimensional representations $X^{(1)}, X^{(2)}, X^{(3)}, X^{(4)}$ are inequivalent, the two-dimensional representations $X^{(5)}$ and $X^{(6)}$ are equivalent.

Tab. 15: Representations of $G_{0,2} = Q_8 = N_2$

	K_1	K_2	K_3	K_4	K_5
g	1	-1	e_1	e_2	e_{12}
$X^{(1)}$	(1)	(1)	(1)	(1)	(1)
$X^{(2)}$	(1)	(1)	(-1)	(1)	(-1)
$X^{(3)}$	(1)	(1)	(1)	(-1)	(-1)
$X^{(4)}$	(1)	(1)	(-1)	(-1)	(1)
$X^{(5)}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} -i & -2i \\ 0 & i \end{pmatrix}$	$\begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}$	$\begin{pmatrix} i & 0 \\ -i & -i \end{pmatrix}$
$X^{(6)}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$

In Table 16, the irreducible representations $X^{(i)}$ of $G_{3,0} = S_1$ are realized in irreducible $G_{3,0}$-invariant submodules of the group algebra $\mathbb{C}[G_{3,0}]$ which is decomposed as follows:

$$\mathbb{C}[G_{3,0}] = \bigoplus_{i=1}^{12} V^{(i)}. \tag{46}$$

The submodules $V^{(i)}$ are spanned by the corresponding vectors $\mathbf{u}_i$, $i = 1, \ldots, 16$, as shown below. The coordinates of these vectors in the standard basis

$$\mathcal{B} = \{1, -1, e_1, -e_1, e_2, -e_2, e_3, -e_3, e_{12}, -e_{12}, e_{13}, -e_{13}, e_{23}, -e_{23}, e_{123}, -e_{123}\}$$

are as follows:

$$
\begin{aligned}
V^{(1)} &= \operatorname{span}\{\mathbf{u}_1\}, & \mathbf{u}_1 &= (1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1),\\
V^{(2)} &= \operatorname{span}\{\mathbf{u}_2\}, & \mathbf{u}_2 &= (-1,-1,-1,-1,-1,-1,1,1,-1,-1,1,1,1,1,1,1),\\
V^{(3)} &= \operatorname{span}\{\mathbf{u}_3\}, & \mathbf{u}_3 &= (-1,-1,-1,-1,1,1,-1,-1,1,1,-1,-1,1,1,1,1),\\
V^{(4)} &= \operatorname{span}\{\mathbf{u}_4\}, & \mathbf{u}_4 &= (1,1,1,1,-1,-1,-1,-1,-1,-1,-1,-1,1,1,1,1),\\
V^{(5)} &= \operatorname{span}\{\mathbf{u}_5\}, & \mathbf{u}_5 &= (-1,-1,1,1,-1,-1,-1,-1,1,1,1,1,-1,-1,1,1),\\
V^{(6)} &= \operatorname{span}\{\mathbf{u}_6\}, & \mathbf{u}_6 &= (1,1,-1,-1,1,1,-1,-1,-1,-1,1,1,-1,-1,1,1),\\
V^{(7)} &= \operatorname{span}\{\mathbf{u}_7\}, & \mathbf{u}_7 &= (1,1,-1,-1,-1,-1,1,1,1,1,-1,-1,-1,-1,1,1),\\
V^{(8)} &= \operatorname{span}\{\mathbf{u}_8\}, & \mathbf{u}_8 &= (-1,-1,1,1,1,1,1,1,-1,-1,-1,-1,-1,-1,1,1),\\
V^{(9)} &= \operatorname{span}\{\mathbf{u}_9,\mathbf{u}_{10}\}, & \mathbf{u}_9 &= (-i,i,0,0,0,0,-i,i,-1,1,0,0,0,0,-1,1),\\
& & \mathbf{u}_{10} &= (0,0,-i,i,-1,1,0,0,0,0,-i,i,-1,1,0,0),\\
V^{(10)} &= \operatorname{span}\{\mathbf{u}_{11},\mathbf{u}_{12}\}, & \mathbf{u}_{11} &= (0,0,0,0,-i,i,-1,1,-i,i,-1,1,0,0,0,0),\\
& & \mathbf{u}_{12} &= (i,-i,-i,i,0,0,0,0,0,0,0,0,1,-1,-1,1),\\
V^{(11)} &= \operatorname{span}\{\mathbf{u}_{13},\mathbf{u}_{14}\}, & \mathbf{u}_{13} &= (0,0,0,0,-i,i,1,-1,i,-i,-1,1,0,0,0,0),\\
& & \mathbf{u}_{14} &= (i,-i,i,-i,0,0,0,0,0,0,0,0,-1,1,-1,1),\\
V^{(12)} &= \operatorname{span}\{\mathbf{u}_{15},\mathbf{u}_{16}\}, & \mathbf{u}_{15} &= (0,0,-i,i,1,-1,0,0,0,0,i,-i,-1,1,0,0),\\
& & \mathbf{u}_{16} &= (-i,i,0,0,0,0,i,-i,1,-1,0,0,0,0,-1,1).
\end{aligned}
\tag{47}
$$

While the representations $X^{(i)}$, $i = 1, \dots, 10$, are all inequivalent and irreducible, the remaining two-dimensional irreducible representations are equivalent as follows: $X^{(12)} \cong X^{(9)}$ and $X^{(11)} \cong X^{(10)}$.

In Table 17, the irreducible representations $X^{(i)}$ of $G_{2,1} = \Omega_1$ are realized in irreducible $G_{2,1}$-invariant submodules of the group algebra $\mathbb{C}[G_{2,1}]$ which is decomposed as follows:

$$\mathbb{C}[G_{2,1}] = \bigoplus_{i=1}^{12} V^{(i)}. \tag{48}$$

The submodules $V^{(i)}$ are spanned by the corresponding vectors $\mathbf{u}_i$, $i = 1, \dots, 16$, as shown below. The coordinates of these vectors in the standard basis

$$\mathcal{B} = \{1, -1, e_1, -e_1, e_2, -e_2, e_3, -e_3, e_{12}, -e_{12}, e_{13}, -e_{13}, e_{23}, -e_{23}, e_{123}, -e_{123}\}$$

Tab. 16: Part 1: Representations of $G_{3,0} = S_1$ for K_i, $i = 1, \ldots, 5$

	K_1	K_2	K_3	K_4	K_5
g	1	-1	e_{123}	$-e_{123}$	e_1
$X^{(1)}$	(1)	(1)	(1)	(1)	(1)
$X^{(2)}$	(1)	(1)	(-1)	(-1)	(1)
$X^{(3)}$	(1)	(1)	(-1)	(-1)	(1)
$X^{(4)}$	(1)	(1)	(1)	(1)	(1)
$X^{(5)}$	(1)	(1)	(-1)	(-1)	(-1)
$X^{(6)}$	(1)	(1)	(1)	(1)	(-1)
$X^{(7)}$	(1)	(1)	(1)	(1)	(-1)
$X^{(8)}$	(1)	(1)	(-1)	(-1)	(-1)
$X^{(9)}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$	$\begin{pmatrix} -i & 0 \\ 0 & -i \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
$X^{(10)}$	$\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} -i & 0 \\ 0 & -i \end{pmatrix}$	$\begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
$X^{(11)}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} -i & 0 \\ 0 & -i \end{pmatrix}$	$\begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$	$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$
$X^{(12)}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$	$\begin{pmatrix} -i & 0 \\ 0 & -i \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

are as follows:

$$
\begin{aligned}
V^{(1)} &= \mathrm{span}\{\mathbf{u}_1\}, & \mathbf{u}_1 &= (1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1),\\
V^{(2)} &= \mathrm{span}\{\mathbf{u}_2\}, & \mathbf{u}_2 &= (-1,-1,-1,-1,-1,-1,1,1,-1,-1,1,1,1,1,1,1),\\
V^{(3)} &= \mathrm{span}\{\mathbf{u}_3\}, & \mathbf{u}_3 &= (-1,-1,-1,-1,1,1,-1,-1,1,1,-1,-1,1,1,1,1),\\
V^{(4)} &= \mathrm{span}\{\mathbf{u}_4\}, & \mathbf{u}_4 &= (1,1,1,1,-1,-1,-1,-1,-1,-1,-1,-1,1,1,1,1),\\
V^{(5)} &= \mathrm{span}\{\mathbf{u}_5\}, & \mathbf{u}_5 &= (-1,-1,1,1,-1,-1,-1,-1,1,1,1,1,-1,-1,1,1),\\
V^{(6)} &= \mathrm{span}\{\mathbf{u}_6\}, & \mathbf{u}_6 &= (1,1,-1,-1,1,1,-1,-1,-1,-1,1,1,-1,-1,1,1),\\
V^{(7)} &= \mathrm{span}\{\mathbf{u}_7\}, & \mathbf{u}_7 &= (1,1,-1,-1,-1,-1,1,1,1,1,-1,-1,-1,-1,1,1),\\
V^{(8)} &= \mathrm{span}\{\mathbf{u}_8\}, & \mathbf{u}_8 &= (-1,-1,1,1,1,1,1,1,-1,-1,-1,-1,-1,-1,1,1),\\
V^{(9)} &= \mathrm{span}\{\mathbf{u}_9,\mathbf{u}_{10}\}, & \mathbf{u}_9 &= (1,-1,1,-1,0,0,0,0,0,0,0,0,-1,1,-1,1),\\
& & \mathbf{u}_{10} &= (0,0,0,0,-1,1,1,-1,1,-1,-1,1,0,0,0,0),\\
V^{(10)} &= \mathrm{span}\{\mathbf{u}_{11},\mathbf{u}_{12}\}, & \mathbf{u}_{11} &= (0,0,0,0,1,-1,1,-1,-1,1,-1,1,0,0,0,0),\\
& & \mathbf{u}_{12} &= (-1,1,-1,1,0,0,0,0,0,0,0,0,-1,1,-1,1),\\
V^{(11)} &= \mathrm{span}\{\mathbf{u}_{13},\mathbf{u}_{14}\}, & \mathbf{u}_{13} &= (0,0,0,0,-1,1,-1,1,-1,1,-1,1,0,0,0,0),\\
& & \mathbf{u}_{14} &= (1,-1,-1,1,0,0,0,0,0,0,0,0,1,-1,-1,1),\\
V^{(12)} &= \mathrm{span}\{\mathbf{u}_{15},\mathbf{u}_{16}\}, & \mathbf{u}_{15} &= (0,0,0,0,1,-1,-1,1,1,-1,-1,1,0,0,0,0),\\
& & \mathbf{u}_{16} &= (-1,1,1,-1,0,0,0,0,0,0,0,0,1,-1,-1,1).
\end{aligned} \tag{49}
$$

Tab. 16: Part 2: Representations of $G_{3,0} = S_1$ for K_i, $i = 6, \dots, 10$

	K_6	K_7	K_8	K_9	K_{10}
g	e_2	e_3	e_{12}	e_{13}	e_{23}
$X^{(1)}$	(1)	(1)	(1)	(1)	(1)
$X^{(2)}$	(1)	(-1)	(1)	(-1)	(-1)
$X^{(3)}$	(-1)	(1)	(-1)	(1)	(-1)
$X^{(4)}$	(-1)	(-1)	(-1)	(-1)	(1)
$X^{(5)}$	(1)	(1)	(-1)	(-1)	(1)
$X^{(6)}$	(1)	(-1)	(-1)	(1)	(-1)
$X^{(7)}$	(-1)	(1)	(1)	(-1)	(-1)
$X^{(8)}$	(-1)	(-1)	(1)	(1)	(1)
$X^{(9)}$	$\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$
$X^{(10)}$	$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}$	$\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$
$X^{(11)}$	$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}$	$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$
$X^{(12)}$	$\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$

While the representations $X^{(i)}$, $i = 1, \dots, 10$, are all inequivalent and irreducible, the remaining two-dimensional irreducible representations are equivalent as follows: $X^{(11)} \cong X^{(9)}$ and $X^{(12)} \cong X^{(10)}$.

In Table 18, the irreducible representations $X^{(i)}$ of $G_{0,3} = \Omega_2$ are realized in irreducible $G_{0,3}$-invariant submodules of the group algebra $\mathbb{C}[G_{0,3}]$ which is decomposed as follows:

$$\mathbb{C}[G_{0,3}] = \bigoplus_{i=1}^{12} V^{(i)}. \tag{50}$$

The submodules $V^{(i)}$ are spanned by the corresponding vectors $\mathbf{u}_i$, $i = 1, \dots, 16$, as shown below. The coordinates of these vectors in the standard basis

$$\mathcal{B} = \{1, -1, e_1, -e_1, e_2, -e_2, e_3, -e_3, e_{12}, -e_{12}, e_{13}, -e_{13}, e_{23}, -e_{23}, e_{123}, -e_{123}\}$$

Tab. 17: Part 1: Representations of $G_{2,1} = \Omega_1$ for K_i, $i = 1, \ldots, 5$

	K_1	K_2	K_3	K_4	K_5
g	1	-1	e_{123}	$-e_{123}$	e_1
$X^{(1)}$	(1)	(1)	(1)	(1)	(1)
$X^{(2)}$	(1)	(1)	(-1)	(-1)	(1)
$X^{(3)}$	(1)	(1)	(-1)	(-1)	(1)
$X^{(4)}$	(1)	(1)	(1)	(1)	(1)
$X^{(5)}$	(1)	(1)	(-1)	(-1)	(-1)
$X^{(6)}$	(1)	(1)	(1)	(1)	(-1)
$X^{(7)}$	(1)	(1)	(1)	(1)	(-1)
$X^{(8)}$	(1)	(1)	(-1)	(-1)	(-1)
$X^{(9)}$	$\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$	$\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}$	$\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}$	$\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$	$\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$
$X^{(10)}$	$\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$	$\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}$	$\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$	$\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}$	$\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}$
$X^{(11)}$	$\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$	$\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}$	$\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}$	$\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$	$\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$
$X^{(12)}$	$\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$	$\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}$	$\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$	$\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}$	$\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$

are as follows:

$$
\begin{aligned}
V^{(1)} &= \operatorname{span}\{\mathbf{u}_1\}, & \mathbf{u}_1 &= (1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1),\\
V^{(2)} &= \operatorname{span}\{\mathbf{u}_2\}, & \mathbf{u}_2 &= (-1,-1,-1,-1,-1,-1,1,1,-1,-1,1,1,1,1,1,1),\\
V^{(3)} &= \operatorname{span}\{\mathbf{u}_3\}, & \mathbf{u}_3 &= (-1,-1,-1,-1,1,1,-1,-1,1,1,-1,-1,1,1,1,1),\\
V^{(4)} &= \operatorname{span}\{\mathbf{u}_4\}, & \mathbf{u}_4 &= (1,1,1,1,-1,-1,-1,-1,-1,-1,-1,-1,1,1,1,1),\\
V^{(5)} &= \operatorname{span}\{\mathbf{u}_5\}, & \mathbf{u}_5 &= (-1,-1,1,1,-1,-1,-1,-1,1,1,1,1,-1,-1,1,1),\\
V^{(6)} &= \operatorname{span}\{\mathbf{u}_6\}, & \mathbf{u}_6 &= (1,1,-1,-1,1,1,-1,-1,-1,-1,1,1,-1,-1,1,1),\\
V^{(7)} &= \operatorname{span}\{\mathbf{u}_7\}, & \mathbf{u}_7 &= (1,1,-1,-1,-1,-1,1,1,1,1,-1,-1,-1,-1,1,1),\\
V^{(8)} &= \operatorname{span}\{\mathbf{u}_8\}, & \mathbf{u}_8 &= (-1,-1,1,1,1,1,1,1,-1,-1,-1,-1,-1,-1,1,1),\\
V^{(9)} &= \operatorname{span}\{\mathbf{u}_9, \mathbf{u}_{10}\}, & \mathbf{u}_9 &= (1,-1,-i,i,0,0,0,0,0,0,0,0,-i,i,-1,1),\\
& & \mathbf{u}_{10} &= (0,0,0,0,1,-1,i,-i,i,-i,-1,1,0,0,0,0),\\
V^{(10)} &= \operatorname{span}\{\mathbf{u}_{11}, \mathbf{u}_{12}\}, & \mathbf{u}_{11} &= (0,0,0,0,i,-i,1,-1,-1,1,i,-i,0,0,0,0),\\
& & \mathbf{u}_{12} &= (-1,1,i,-i,0,0,0,0,0,0,0,0,-i,i,-1,1),\\
V^{(11)} &= \operatorname{span}\{\mathbf{u}_{13}, \mathbf{u}_{14}\}, & \mathbf{u}_{13} &= (0,0,0,0,1,-1,-i,i,-i,i,-1,1,0,0,0,0),\\
& & \mathbf{u}_{14} &= (1,-1,i,-i,0,0,0,0,0,0,0,0,i,-i,-1,1),\\
V^{(12)} &= \operatorname{span}\{\mathbf{u}_{15}, \mathbf{u}_{16}\}, & \mathbf{u}_{15} &= (0,0,0,0,-1,1,-i,i,i,-i,-1,1,0,0,0,0),\\
(51) & & \mathbf{u}_{16} &= (-1,1,-i,i,0,0,0,0,0,0,0,0,i,-i,-1,1).
\end{aligned}
$$

Tab. 17: Part 2: Representations of $G_{2,1} = \Omega_1$ for K_i, $i = 6, \dots, 10$

	K_6	K_7	K_8	K_9	K_{10}
g	e_2	e_3	e_{12}	e_{13}	e_{23}
$X^{(1)}$	$\begin{pmatrix}1\end{pmatrix}$	$\begin{pmatrix}1\end{pmatrix}$	$\begin{pmatrix}1\end{pmatrix}$	$\begin{pmatrix}1\end{pmatrix}$	$\begin{pmatrix}1\end{pmatrix}$
$X^{(2)}$	$\begin{pmatrix}1\end{pmatrix}$	$\begin{pmatrix}-1\end{pmatrix}$	$\begin{pmatrix}1\end{pmatrix}$	$\begin{pmatrix}-1\end{pmatrix}$	$\begin{pmatrix}-1\end{pmatrix}$
$X^{(3)}$	$\begin{pmatrix}-1\end{pmatrix}$	$\begin{pmatrix}1\end{pmatrix}$	$\begin{pmatrix}-1\end{pmatrix}$	$\begin{pmatrix}1\end{pmatrix}$	$\begin{pmatrix}-1\end{pmatrix}$
$X^{(4)}$	$\begin{pmatrix}-1\end{pmatrix}$	$\begin{pmatrix}-1\end{pmatrix}$	$\begin{pmatrix}-1\end{pmatrix}$	$\begin{pmatrix}-1\end{pmatrix}$	$\begin{pmatrix}1\end{pmatrix}$
$X^{(5)}$	$\begin{pmatrix}1\end{pmatrix}$	$\begin{pmatrix}1\end{pmatrix}$	$\begin{pmatrix}-1\end{pmatrix}$	$\begin{pmatrix}-1\end{pmatrix}$	$\begin{pmatrix}1\end{pmatrix}$
$X^{(6)}$	$\begin{pmatrix}1\end{pmatrix}$	$\begin{pmatrix}-1\end{pmatrix}$	$\begin{pmatrix}-1\end{pmatrix}$	$\begin{pmatrix}1\end{pmatrix}$	$\begin{pmatrix}-1\end{pmatrix}$
$X^{(7)}$	$\begin{pmatrix}-1\end{pmatrix}$	$\begin{pmatrix}1\end{pmatrix}$	$\begin{pmatrix}1\end{pmatrix}$	$\begin{pmatrix}-1\end{pmatrix}$	$\begin{pmatrix}-1\end{pmatrix}$
$X^{(8)}$	$\begin{pmatrix}-1\end{pmatrix}$	$\begin{pmatrix}-1\end{pmatrix}$	$\begin{pmatrix}1\end{pmatrix}$	$\begin{pmatrix}1\end{pmatrix}$	$\begin{pmatrix}1\end{pmatrix}$
$X^{(9)}$	$\begin{pmatrix}0 & -1\\ -1 & 0\end{pmatrix}$	$\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$	$\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$	$\begin{pmatrix}0 & -1\\ -1 & 0\end{pmatrix}$	$\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}$
$X^{(10)}$	$\begin{pmatrix}0 & -1\\ -1 & 0\end{pmatrix}$	$\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$	$\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$	$\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$	$\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}$
$X^{(11)}$	$\begin{pmatrix}0 & -1\\ -1 & 0\end{pmatrix}$	$\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$	$\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$	$\begin{pmatrix}0 & -1\\ -1 & 0\end{pmatrix}$	$\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}$
$X^{(12)}$	$\begin{pmatrix}0 & -1\\ -1 & 0\end{pmatrix}$	$\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$	$\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$	$\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$	$\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$

While the representations $X^{(i)}$, $i = 1, \dots, 10$, are all inequivalent and irreducible, the remaining two-dimensional irreducible representations are equivalent as follows: $X^{(12)} \cong X^{(10)}$ and $X^{(11)} \cong X^{(9)}$.

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Tab. 18: Part 1: Representations of $G_{0,3} = \Omega_2$ for K_i, $i = 1, \ldots, 5$

	K_1	K_2	K_3	K_4	K_5
g	1	-1	e_{123}	$-e_{123}$	e_1
$X^{(1)}$	(1)	(1)	(1)	(1)	(1)
$X^{(2)}$	(1)	(1)	(-1)	(-1)	(1)
$X^{(3)}$	(1)	(1)	(-1)	(-1)	(1)
$X^{(4)}$	(1)	(1)	(1)	(1)	(1)
$X^{(5)}$	(1)	(1)	(-1)	(-1)	(-1)
$X^{(6)}$	(1)	(1)	(1)	(1)	(-1)
$X^{(7)}$	(1)	(1)	(1)	(1)	(-1)
$X^{(8)}$	(1)	(1)	(-1)	(-1)	(-1)
$X^{(9)}$	$\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$	$\begin{pmatrix}-1 & 0\\0 & -1\end{pmatrix}$	$\begin{pmatrix}-1 & 0\\0 & -1\end{pmatrix}$	$\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$	$\begin{pmatrix}i & 0\\0 & -i\end{pmatrix}$
$X^{(10)}$	$\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$	$\begin{pmatrix}-1 & 0\\0 & -1\end{pmatrix}$	$\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$	$\begin{pmatrix}-1 & 0\\0 & -1\end{pmatrix}$	$\begin{pmatrix}-i & 0\\0 & i\end{pmatrix}$
$X^{(11)}$	$\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$	$\begin{pmatrix}-1 & 0\\0 & -1\end{pmatrix}$	$\begin{pmatrix}-1 & 0\\0 & -1\end{pmatrix}$	$\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$	$\begin{pmatrix}i & 0\\0 & -i\end{pmatrix}$
$X^{(12)}$	$\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$	$\begin{pmatrix}-1 & 0\\0 & -1\end{pmatrix}$	$\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$	$\begin{pmatrix}-1 & 0\\0 & -1\end{pmatrix}$	$\begin{pmatrix}i & 0\\0 & -i\end{pmatrix}$

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Tab. 18: Part 2: Representations of $G_{0,3} = \Omega_2$ for K_i, $i = 6, \ldots, 10$

	K_6	K_7	K_8	K_9	K_{10}
g	e_2	e_3	e_{12}	e_{13}	e_{23}
$X^{(1)}$	$\begin{pmatrix}1\end{pmatrix}$	$\begin{pmatrix}1\end{pmatrix}$	$\begin{pmatrix}1\end{pmatrix}$	$\begin{pmatrix}1\end{pmatrix}$	$\begin{pmatrix}1\end{pmatrix}$
$X^{(2)}$	$\begin{pmatrix}1\end{pmatrix}$	$\begin{pmatrix}-1\end{pmatrix}$	$\begin{pmatrix}1\end{pmatrix}$	$\begin{pmatrix}-1\end{pmatrix}$	$\begin{pmatrix}-1\end{pmatrix}$
$X^{(3)}$	$\begin{pmatrix}-1\end{pmatrix}$	$\begin{pmatrix}1\end{pmatrix}$	$\begin{pmatrix}-1\end{pmatrix}$	$\begin{pmatrix}1\end{pmatrix}$	$\begin{pmatrix}-1\end{pmatrix}$
$X^{(4)}$	$\begin{pmatrix}-1\end{pmatrix}$	$\begin{pmatrix}-1\end{pmatrix}$	$\begin{pmatrix}-1\end{pmatrix}$	$\begin{pmatrix}-1\end{pmatrix}$	$\begin{pmatrix}1\end{pmatrix}$
$X^{(5)}$	$\begin{pmatrix}1\end{pmatrix}$	$\begin{pmatrix}1\end{pmatrix}$	$\begin{pmatrix}-1\end{pmatrix}$	$\begin{pmatrix}-1\end{pmatrix}$	$\begin{pmatrix}1\end{pmatrix}$
$X^{(6)}$	$\begin{pmatrix}1\end{pmatrix}$	$\begin{pmatrix}-1\end{pmatrix}$	$\begin{pmatrix}-1\end{pmatrix}$	$\begin{pmatrix}1\end{pmatrix}$	$\begin{pmatrix}-1\end{pmatrix}$
$X^{(7)}$	$\begin{pmatrix}-1\end{pmatrix}$	$\begin{pmatrix}1\end{pmatrix}$	$\begin{pmatrix}1\end{pmatrix}$	$\begin{pmatrix}-1\end{pmatrix}$	$\begin{pmatrix}-1\end{pmatrix}$
$X^{(8)}$	$\begin{pmatrix}-1\end{pmatrix}$	$\begin{pmatrix}-1\end{pmatrix}$	$\begin{pmatrix}1\end{pmatrix}$	$\begin{pmatrix}1\end{pmatrix}$	$\begin{pmatrix}1\end{pmatrix}$
$X^{(9)}$	$\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$	$\begin{pmatrix}0 & -i\\ -i & 0\end{pmatrix}$	$\begin{pmatrix}0 & -i\\ -i & 0\end{pmatrix}$	$\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$	$\begin{pmatrix}i & 0\\ 0 & -i\end{pmatrix}$
$X^{(10)}$	$\begin{pmatrix}0 & i\\ i & 0\end{pmatrix}$	$\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$	$\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$	$\begin{pmatrix}0 & i\\ i & 0\end{pmatrix}$	$\begin{pmatrix}i & 0\\ 0 & -i\end{pmatrix}$
$X^{(11)}$	$\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$	$\begin{pmatrix}0 & i\\ i & 0\end{pmatrix}$	$\begin{pmatrix}0 & i\\ i & 0\end{pmatrix}$	$\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$	$\begin{pmatrix}i & 0\\ 0 & -i\end{pmatrix}$
$X^{(12)}$	$\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$	$\begin{pmatrix}0 & -i\\ -i & 0\end{pmatrix}$	$\begin{pmatrix}0 & i\\ i & 0\end{pmatrix}$	$\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$	$\begin{pmatrix}-i & 0\\ 0 & i\end{pmatrix}$

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