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ON THE STRUCTURE THEOREM OF CLIFFORD ALGEBRAS

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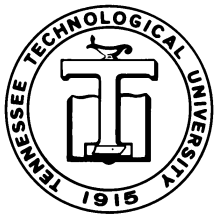

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ON THE STRUCTURE THEOREM OF CLIFFORD ALGEBRAS

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Abstract. In this paper, theory and construction of spinor representations of real Clifford algebras $C\ell_{p,q}$ in minimal left ideals are reviewed. Connection with a general theory of semisimple rings is shown. The actual computations can be found in, for example, [2].

Keywords. Artinian ring, Clifford algebra, division ring, group algebra, idempotent, minimal left ideal, semisimple module, Radon-Hurwitz number, semisimple ring, Wedderburn-Artin Theorem

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1. Introduction

Theory of spinor representations of real Clifford algebras $C\ell_{p,q}$ over a quadratic space (V, Q) with a nondegenerate quadratic form Q of signature (p, q) is well known [11, 18, 19, 24]. The purpose of this paper is to review the structure theorem of these algebras in the context of a general theory of semisimple rings culminating with Wedderburn-Artin Theorem [26].

Section 2 is devoted to a short review of general background material on the theory of semisimple rings and modules as a generalization of the representation theory of group algebras of finite groups [17, 26]. While it is well-known that Clifford algebras $C\ell_{p,q}$ are associative finite-dimensional unital semisimple $\mathbb{R}$-algebras, hence the representation theory of semisimple rings [26, Chapter 7] applies to them, it is also possible to view these algebras

as twisted group algebras $\mathbb{R}^t[(\mathbb{Z}_2)^n]$ of a finite group $(\mathbb{Z}_2)^n$ [5–7, 9, 13, 23]. While this last approach is not pursued here, for a connection between Clifford algebras $C\ell_{p,q}$ and finite groups, see [1, 6, 7, 10, 20, 21, 27] and references therein.

In Section 3, we state the main Structure Theorem on Clifford algebras $C\ell_{p,q}$ and relate it to the general theory of semisimple rings, especially to the Wedderburn-Artin theorem. For details of computation of spinor representations, we refer to [2] where these computations were done in great detail by hand and by using `CLIFFORD`, a Maple package specifically designed for computing and storing spinor representations of Clifford algebras $C\ell_{p,q}$ for $n = p + q \leq 9$ [3, 4].

Our standard references on the theory of modules, semisimple rings and their representation is [26]; for Clifford algebras we use [11, 18, 19] and references therein; on representation theory of finite groups we refer to [17, 25] and for the group theory we refer to [12, 14, 22, 26].

2. Introduction to Semisimple Rings and Modules

This brief introduction to the theory of semisimple rings is based on [26, Chapter 7] and it is stated in the language of left R-modules. Here, R denotes an associative ring with unity 1. We omit proofs as they can be found in Rotman [26].

Definition 1. Let R be a ring. A **left R-module** is an additive abelian group M equipped with **scalar multiplication** $R \times M \to M$, denoted $(r, m) \mapsto rm$, such that the following axioms hold for all $m, m' \in M$ and all $r, r' \in R$:

(i) $r(m + m') = rm + rm'$,
(ii) $(r + r')m = rm + r'm$,
(iii) $(rr')m = r(r'm)$,
(iv) $1m = m$.

Left R-modules are often denoted by ${}_RM$.

In a similar manner one can define a **right R-module** with the action by the ring elements on M from the right. When R and S are rings and M is an abelian group, then M is a (R, S)**-bimodule**, denoted by ${}_RM_S$, if M is a left R-module, a right S-module, and the two scalar multiplications are related by an associative law: $r(ms) = (rm)s$ for all $r \in R, m \in M$, and $s \in S$.

We recall that a spinor left ideal S in a simple Clifford algebra $C\ell_{p,q}$ by definition carries an irreducible and faithful representation of the algebra, and it is defined as $C\ell_{p,q}f$ where f is a primitive idempotent in $C\ell_{p,q}$. Thus, as it is known from the Structure Theorem (see Section 3), that these ideals are (R, S)-bimodules where $R = C\ell_{p,q}$ and $S = fC\ell_{p,q}f$. Similarly, the right spinor modules $fC\ell_{p,q}$ are (S, R)-bimodules. Notice that the associative law mentioned above is automatically satisfied because $C\ell_{p,q}$ is associative.

We just recall that when k is a field, every finite-dimensional k-algebra A is both **left** and **right noetherian**, that is, any ascending chain of left and right ideals stops (the **ACC ascending chain condition**). This is important for Clifford algebras because, eventually, we will see that every Clifford algebra can be decomposed into a finite direct sum of left spinor $C\ell_{p,q}$-modules (ideals). For completeness we mention that every finite-dimensional

k-algebra A is both **left** and **right artinian**, that is, any descending chain of left and right ideals stops (the **DCC ascending chain condition**).

Thus, every Clifford algebra $C\ell_{p,q}$, as well as every group algebra kG, when G is a finite group, which then makes kG finite dimensional, have both chain conditions by a dimensionality argument.

Definition 2. A left ideal L in a ring R is a **minimal left ideal** if $L \neq (0)$ and there is no left ideal J with $(0) \subsetneq J \subsetneq L$.

One standard example of minimal left ideals in matrix algebras $R = \mathrm{Mat}(n, k)$ are the subspaces $\mathrm{COL}(j)$, $1 \leq j \leq n$, of $\mathrm{Mat}(n, k)$ consisting of matrices $[a_{i,j}]$ such that $a_{i,k} = 0$ when $k \neq j$ (cf. [26, Example 7.9]).

The following proposition relates minimal left ideals in a ring R to simple left R-modules. Recall that a left R-module M is **simple** (or **irreducible**) if $M \neq \{0\}$ and M has no proper nonzero submodules.

Proposition 1 (Rotman [26])**.**

(i) *Every minimal left ideal L in a ring R is a simple left R-module.*
(ii) *If R is left artinian, then every nonzero left ideal I contains a minimal left ideal.*

Thus, the above proposition applies to Clifford algebras $C\ell_{p,q}$: every left spinor ideal S in $C\ell_{p,q}$ is a simple left $C\ell_{p,q}$-module; and, every left ideal in $C\ell_{p,q}$ contains a spinor ideal.

Recall that if D is a division ring, then a left (or right) D-module V is called a **left** (or **right**) **vector space** over D. In particular, when the division ring is a field k, then we have a familiar concept of a k-vector space. Since the concept of linear independence of vectors generalizes from k-vector spaces to D-vector spaces, we have the following result.

Proposition 2 (Rotman [26])**.** *Let V be a finitely generated*[1] *left vector space over a division ring D.*

(i) *V is a direct sum of copies of D; that is, every finitely generated left vector space over D has a basis.*
(ii) *Any two bases of V have the same number of elements.*

Since we know from the Structure Theorem, that every spinor left ideal S in simple Clifford algebras $C\ell_{p,q}$ ($p - q \neq 1 \bmod 4$) is a right $\mathbb{K}$-module where $\mathbb{K}$ is one of the division rings $\mathbb{R}, \mathbb{C}$, or $\mathbb{H}$, the above proposition simply tells us that every spinor left ideal S is finite-dimensional over $\mathbb{K}$ where $\mathbb{K}$ is one of $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$.

In semisimple Clifford algebras $C\ell_{p,q}$ ($p - q = 1 \bmod 4$), we have to be careful as the faithful double spinor representations are realized in the direct sum of two spinor ideals $S \oplus \hat{S}$ which are right $\mathbb{K} \oplus \hat{\mathbb{K}}$-modules, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{H}$.[2] Yet, it is easy to show that $\mathbb{K} \oplus \hat{\mathbb{K}}$ is not a division ring.

[1]The term "finitely generated" means that every vector in V is a linear combination of a finite number of certain vectors $\{x_1, \ldots, x_n\}$ with coefficients from R. In particular, a k-vector space is finitely generated if and only if it is finite-dimensional [26, Page 405].

[2]Here, $\hat{S} = \{\hat{\psi} \mid \in S\}$, and similarly for $\hat{\mathbb{K}}$, where $\hat{}$ denotes the grade involution in $C\ell_{p,q}$.

Thus, Proposition 2 tells us that every finitely generated left (or right) vector space V over a division ring D has a left (a right) dimension, which may be denoted $\dim V$. In [16] Jacobson gives an example of a division ring D and an abelian group V, which is both a right and a left D-vector space, such that the left and the right dimensions are not equal. In our discussion, spinor minimal ideal S will always be a left $C\ell_{p,q}$-module and a right $\mathbb{K}$-module.

Since semisimple rings generalize the concept of a group algebra $\mathbb{C}G$ for a finite group G (cf. [17, 26]), we first discuss semisimple modules over a ring R.

Definition 3. A left R-module is **semisimple** if it is a direct sum of (possibly infinitely many) simple modules.

The following result is an important characterization of semisimple modules.

Proposition 3 (Rotman [26])**.** *A left R-module M over a ring R is semisimple if and only if every submodule of M is a direct summand.*

Recall that if a ring R is viewed as a left R-module, then its submodules are its left ideals, and, a left ideal is minimal if and only if it is a simple left R-module [26].

Definition 4. A ring R is **left semisimple**[3] if it is a direct sum of minimal left ideals.

One of the important consequences of the above for the theory of Clifford algebras, is the following proposition.

Proposition 4 (Rotman [26])**.** *Let R be a left semisimple ring.*

(i) *R is a direct sum of finitely many minimal left ideals.*
(ii) *R has both chain conditions on left ideals.*

From a proof of the above proposition one learns that, while $R = \bigoplus_i L_i$, that is, R is a direct sum of finitely-many left minimal ideals, the unity 1 decomposes into a sum $1 = \sum_i f_i$ of mutually annihilating primitive idempotents f_i, that is, $(f_i)^2 = f_i$, and $f_i f_j = f_j f_i = 0, i \neq j$. Furthermore, we find that $L_i = Rf_i$ for every i.

We can conclude from the following fundamental result [15, 26] that every Clifford algebra $C\ell_{p,q}$ is a semisimple ring, because every Clifford algebra is a twisted group algebra $\mathbb{R}^t[(\mathbb{Z}_2)^n]$ for $n = p + q$ and a suitable twist [1, 7, 9].

Theorem 1 (Maschke's Theorem)**.** *If G is a finite group and k is a field whose characteristic does not divide $|G|$, the kG is a left semisimple ring.*

For characterizations of left semisimple rings, we refer to [26, Section 7.3].

Before stating Wedderburn-Artin Theorem, which is all-important to the theory of Clifford algebras, we conclude this part with a definition and two propositions.

Definition 5. A ring R is **simple** if it is not the zero ring and it has no proper nonzero two-sided ideals.

Proposition 5 (Rotman [26])**.** *If D is a division ring, then $R = \mathrm{Mat}(n, D)$ is a simple ring.*

[3]One can define a **right semisimple** ring R if it is a direct sum of minimal right ideals. However, it is known [26, Corollary 7.45] that a ring is left semisimple if and only if it is right semisimple.

Proposition 6 (Rotman [26]). *If $R = \bigoplus_j L_j$ is a left semisimple ring, where the L_j are minimal left ideals, then every simple R-module S is isomorphic to L_j for some j.*

The main consequence of this last result is that every simple, hence irreducible, left $C\ell_{p,q}$-module, that is, every (left) spinor module of $C\ell_{p,q}$, is isomorphic to some minimal left ideal L_j in the direct sum decomposition of $R = C\ell_{p,q}$.

Following Rotman, we divide the Wedderburn-Artin Theorem into the existence part and a uniqueness part. We also remark after Rotman that Wedderburn proved the existence theorem 2 for semisimple k-algebras, where k is a field, while E. Artin generalized this result to what is now known as the Wedderburn-Artin Theorem.

Theorem 2 (Wedderburn-Artin I). *A ring R is left semisimple if and only if R is isomorphic to a direct product of matrix rings over division rings $D_1, \ldots, D_m$, that is*

$$R \cong \mathrm{Mat}(n_1, D_1) \times \cdots \times \mathrm{Mat}(n_m, D_m). \tag{1}$$

A proof of the above theorem yields that if $R = \bigoplus_j L_j$ as in Proposition 6, then each division ring $D_j = \mathrm{End}_R(L_j), j = 1, \ldots, m$, where $\mathrm{End}_R(L_j)$ denotes the ring of all R-endomorphisms of L_j. Another consequence is the following corollary.

Corollary 1. *A ring R is left semisimple if and only if it is right semisimple.*

Thus, we may refer to a ring as being **semisimple** without specifying from which side.[4] However, we have the following result which we know applies to Clifford algebras $C\ell_{p,q}$. More importantly, its corollary explains part of the Structure Theorem which applies to simple Clifford algebras. Recall from the above that every Clifford algebra $C\ell_{p,q}$ is left artinian (because it is finite-dimensional).

Proposition 7 (Rotman [26]). *A simple left artinian ring R is semisimple.*

Corollary 2. *If A is a simple left artinian ring, then $A \cong \mathrm{Mat}(n, D)$ for some $n \geq 1$ and some division ring D.*

Before we conclude this section with the second part of the Wedderburn-Artin Theorem, which gives certain uniqueness of the decomposition (1), we state the following definition and a lemma.

Definition 6. Let R be a left semisimple ring, and let

$$R = L_1 \oplus \cdots \oplus L_n, \tag{2}$$

where the L_j are minimal left ideals. Let the ideals $L_1, \ldots, L_m$, possibly after re-indexing, be such that no two among them are isomorphic, and so that every L_j in the given decomposition of R is isomorphic to one and only one L_i for $1 \leq i \leq m$. The left ideals

$$B_i = \bigoplus_{L_j \cong L_i} L_j \tag{3}$$

are called the **simple components** of R relative to the decomposition $R = \bigoplus_j L_j$.

[4]Not every simple ring is semisimple, cf. [26, Page 554] and reference therein.

Lemma 1 (Rotman [26]). *Let R be a semisimple ring, and let*

$$R = L_1 \oplus \cdots \oplus L_n = B_1 \oplus \cdots \oplus B_m \tag{4}$$

where the L_j are minimal left ideals and the B_i are the corresponding simple components of R.

(i) *Each B_i is a ring that is also a two-sided ideal in R, and $B_iB_j = (0)$ if $i \neq j$.*
(ii) *If L is any minimal left ideal in R, not necessarily occurring in the given decomposition of R, then $L \cong L_i$ for some i and $L \subseteq B_i$.*
(iii) *Every two-sided ideal in R is a direct sum of simple components.*
(iv) *Each B_i is a simple ring.*

Thus, we will gather from the Structure Theorem, that for simple Clifford algebras $C\ell_{p,q}$ we have only one simple component, hence $m = 1$, and thus all 2^k left minimal ideals generated by a complete set of 2^k primitive mutually annihilating idempotents which provide an orthogonal decomposition of the unity 1 in $C\ell_{p,q}$ (see part (c) of the theorem and notation therein). Then, for semisimple Clifford algebras $C\ell_{p,q}$ we have obviously $m = 2$.

Furthermore, we have the following corollary results.

Corollary 3 (Rotman [26]).

(1) *The simple components $B_1, \ldots, B_m$ of a semisimple ring R do not depend on a decomposition of R as a direct sum of minimal left ideals;*
(2) *Let A be a simple artinian ring. Then,*
   (i) *$A \cong \mathrm{Mat}(n, D)$ for some division ring D. If L is a minimal left ideal in A, then every simple left A-module is isomorphic to L; moreover, $D^{\mathrm{op}} \cong \mathrm{End}_A(L)$.*[5]
   (ii) *Two finitely generated left A-modules M and N are isomorphic if and only if $\dim_D(M) = \dim_D(N)$.*

As we can see, part (1) of this last corollary gives a certain invariance in the decomposition of a semisimple ring into a direct sum of simple components. Part (2i), for the left artinian Clifford algebras $C\ell_{p,q}$ implies that simple Clifford algebras ($p - q \neq 1 \bmod 4$) are simple algebras isomorphic to a matrix algebra over a suitable division ring D. From the Structure Theorem we know that D is one of $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$, depending on the value of $p - q \bmod 8$. Part (2ii) tells us that any two spinor ideals S and S', which are simple right $\mathbb{K}$-modules (due the right action of the division ring $\mathbb{K} = fC\ell_{p,q}f$ on each of them) are isomorphic since their dimensions over $\mathbb{K}$ are the same.

We conclude this introduction to the theory of semisimple rings with the following uniqueness theorem.

Theorem 3 (Wedderburn-Artin II). *Every semisimple ring R is a direct product,*

$$R \cong \mathrm{Mat}(n_1, D_1) \times \cdots \times \mathrm{Mat}(n_m, D_m), \tag{5}$$

where $n_i \geq 1$, and D_i is a division ring, and the numbers m and n_i, as well as the division rings D_i, are uniquely determined by R.

[5]By D^{op} we mean the **opposite ring** of D: It is defined as $D^{\mathrm{op}} = \{a^{\mathrm{op}} \mid a \in D\}$ with multiplication defined as $a^{\mathrm{op}} \cdot b^{\mathrm{op}} = (ba)^{\mathrm{op}}$.

Thus, the above results, and especially the Wedderburn-Artin Theorem (parts I and II), shed a new light on the main Structure Theorem given in the following section. In particular, we see it as a special case of the theory of semisimple rings, including the left artinian rings, applied to the finite dimensional Clifford algebras $C\ell_{p,q}$.

We remark that the above theory applies to the group algebras kG where k is an algebraically closed field and G is a finite group.

3. The Main Structure Theorem on Real Clifford Algebras $C\ell_{p,q}$

We have the following main theorem that describes the structure of Clifford algebras $C\ell_{p,q}$ and their spinorial representations. In the following, we will analyze statements in that theorem. The same information is encoded in the well–known Table 1 in [19, Page 217].

Structure Theorem. *Let $C\ell_{p,q}$ be the universal real Clifford algebra over (V, Q), Q is non-degenerate of signature (p, q).*

(a) *When $p - q \neq 1 \bmod 4$ then $C\ell_{p,q}$ is a simple algebra of dimension 2^{p+q} isomorphic with a full matrix algebra $\mathrm{Mat}(2^k, \mathbb{K})$ over a division ring $\mathbb{K}$ where $k = q - r_{q-p}$ and r_i is the Radon-Hurwitz number.*[6] *Here $\mathbb{K}$ is one of $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$ when $(p-q) \bmod 8$ is $0, 2$, or $3, 7$, or $4, 6$.*

(b) *When $p-q = 1 \bmod 4$ then $C\ell_{p,q}$ is a semisimple algebra of dimension 2^{p+q} isomorphic to $\mathrm{Mat}(2^{k-1}, \mathbb{K}) \oplus \mathrm{Mat}(2^{k-1}, \mathbb{K})$, $k = q - r_{q-p}$, and $\mathbb{K}$ is isomorphic to $\mathbb{R}$ or $\mathbb{H}$ depending whether $(p-q) \bmod 8$ is 1 or 5. Each of the two simple direct components of $C\ell_{p,q}$ is projected out by one of the two central idempotents $\frac{1}{2}(1 \pm \mathbf{e}_{12\ldots n})$.*

(c) *Any element f in $C\ell_{p,q}$ expressible as a product*

$$f = \frac{1}{2}(1 \pm \mathbf{e}_{\underline{i}_1})\frac{1}{2}(1 \pm \mathbf{e}_{\underline{i}_2}) \cdots \frac{1}{2}(1 \pm \mathbf{e}_{\underline{i}_k}) \tag{6}$$

where $\mathbf{e}_{\underline{i}_j}$, $j = 1, \ldots, k$, are commuting basis monomials in $\mathcal{B}$ with square 1 and $k = q - r_{q-p}$ generating a group of order 2^k, is a primitive idempotent in $C\ell_{p,q}$. Furthermore, $C\ell_{p,q}$ has a complete set of 2^k such primitive mutually annihilating idempotents which add up to the unity 1 of $C\ell_{p,q}$.

(d) *When $(p-q) \bmod 8$ is $0, 1, 2$, or $3, 7$, or $4, 5, 6$, then the division ring $\mathbb{K} = fC\ell_{p,q}f$ is isomorphic to $\mathbb{R}$ or $\mathbb{C}$ or $\mathbb{H}$, and the map $S \times \mathbb{K} \to S$, $(\psi, \lambda) \mapsto \psi\lambda$ defines a right $\mathbb{K}$-module structure on the minimal left ideal $S = C\ell_{p,q}f$.*

(e) *When $C\ell_{p,q}$ is simple, then the map*

$$C\ell_{p,q} \xrightarrow{\gamma} \mathrm{End}_{\mathbb{K}}(S), \quad u \mapsto \gamma(u), \quad \gamma(u)\psi = u\psi \tag{7}$$

gives an irreducible and faithful representation of $C\ell_{p,q}$ in S.

(f) *When $C\ell_{p,q}$ is semisimple, then the map*

$$C\ell_{p,q} \xrightarrow{\gamma} \mathrm{End}_{\mathbb{K}\oplus\hat{\mathbb{K}}}(S \oplus \hat{S}), \quad u \mapsto \gamma(u), \quad \gamma(u)\psi = u\psi \tag{8}$$

gives a faithful but reducible representation of $C\ell_{p,q}$ in the double spinor space $S \oplus \hat{S}$ where $S = \{uf \,|\, u \in C\ell_{p,q}\}$, $\hat{S} = \{u\hat{f} \,|\, u \in C\ell_{p,q}\}$ and $\hat{}$ stands for the grade involution in $C\ell_{p,q}$. In this case, the ideal $S \oplus \hat{S}$ is a right $\mathbb{K} \oplus \hat{\mathbb{K}}$-module structure,

[6] The Radon-Hurwitz number is defined by recursion as $r_{i+8} = r_i + 4$ and these initial values: $r_0 = 0$, $r_1 = 1$, $r_2 = r_3 = 2, r_4 = r_5 = r_6 = r_7 = 3$.

$\hat{\mathbb{K}} = \{\hat{\lambda} \,|\, \lambda \in \mathbb{K}\}$, *and* $\mathbb{K} \oplus \hat{\mathbb{K}}$ *is isomorphic to* $\mathbb{R} \oplus \mathbb{R}$ *when* $p - q = 1 \bmod 8$ *or to* $\mathbb{H} \oplus \hat{\mathbb{H}}$ *when* $p - q = 5 \bmod 8$.

Parts (a) and (b) address simple and semisimple Clifford algebras $C\ell_{p,q}$ which are distinguished by the value of $p - q \bmod 4$ while the dimension of $C\ell_{p,q}$ is 2^{p+q}. For simple algebras, the Radon-Hurwitz number r_i defined recursively as shown, determines the value of the exponent $k = q - r_{q-p}$ such that

$$C\ell_{p,q} \cong \mathrm{Mat}(2^k, \mathbb{K}) \quad \text{when } p - q \neq 1 \bmod 4. \tag{9}$$

Then, the value of $p-q \bmod 8$ ("Periodicity of Eight" cf. [8,19]) determines whether $\mathbb{K} \cong \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$. Furthermore, this automatically tells us, based on the theory outlined above, that

$$C\ell_{p,q} = L_1 \oplus \cdots \oplus L_N, \quad N = 2^k, \tag{10}$$

that is, that the Clifford algebras decomposes into a direct sum of $N = 2^k$ minimal left ideals (simple left $C\ell_{p,q}$-modules) L_i, each of which is generated by a primitive idempotent. How to find these primitive mutually annihilating idempotents, is determined in Part (c).

In Part (b) we also learn that the Clifford algebra $C\ell_{p,q}$ is semisimple as it is the direct sum of two simple algebras:

$$C\ell_{p,q} \cong \mathrm{Mat}(2^{k-1}, \mathbb{K}) \oplus \mathrm{Mat}(2^{k-1}, \mathbb{K}) \quad \text{when } p - q = 1 \bmod 4. \tag{11}$$

Thus, we have two simple components in the algebra, each of which is a subalgebra. Notice that the two algebra elements

$$c_1 = \frac{1}{2}(1 + \mathbf{e}_{12\ldots n}) \quad \text{and} \quad c_2 = \frac{1}{2}(1 - \mathbf{e}_{12\ldots n}) \tag{12}$$

are central, that is, each belongs to the center $Z(C\ell_{p,q})$ of the algebra.[7] This requires that $n = p+q$ be odd, so that the unit pseudoscalar $\mathbf{e}_{12\ldots n}$ would commute with each generator $\mathbf{e}_i$, and that $(\mathbf{e}_{12\ldots n})^2 = 1$, so that expressions (12) would truly be idempotents. Notice, that the idempotents c_1, c_2 provide an **orthogonal decomposition** of the unity 1 since $c_1 + c_2 = 1$, and they are mutually annihilating since $c_1 c_2 = c_2 c_1 = 0$. Thus,

$$C\ell_{p,q} = C\ell_{p,q} c_1 \oplus C\ell_{p,q} c_2 \tag{13}$$

where each $C\ell_{p,q} c_i$ is a simple subalgebra of $C\ell_{p,q}$. Hence, by Part (a), each subalgebra is isomorphic to $\mathrm{Mat}(2^{k-1}, \mathbb{K})$ where $\mathbb{K}$ is either $\mathbb{R}$ or $\mathbb{H}$ depending on the value of $p-q \bmod 8$, as indicated.

Part (c) tells us how to find a complete set of 2^k primitive mutually annihilating idempotents, obtained by independently varying signs $\pm$ in each factor in (6), provide an orthogonal decomposition of the unity. The set of k commuting basis monomials $\mathbf{e}_{\underline{i}_1}, \ldots, \mathbf{e}_{\underline{i}_k}$, which square to 1, is not unique. Stabilizer groups of these 2^k primitive idempotents $f_1, \ldots, f_N$ ($N = 2^k$) under the conjugate action of Salingaros vee groups are discussed in [6, 7]. It should be remarked, that each idempotent in (6) must have exactly k factors in order to be primitive.

Thus, we conclude from Part (c) that

$$C\ell_{p,q} = C\ell_{p,q} f_1 \oplus \cdots \oplus C\ell_{p,q} f_N, \quad N = 2^k, \tag{14}$$

[7]The center $Z(A)$ of an k-algebra A contains all elements in A which commute with every element in A. In particular, from the definition of the k-algebra, $\lambda 1 \in Z(C\ell_{p,q})$ for every $\lambda \in k$.

is a decomposition of the Clifford algebra $C\ell_{p,q}$ into a direct sum of minimal left ideals, or, simple left $C\ell_{p,q}$-modules.

Part (d) determines the unique division ring $\mathbb{K} = fC\ell_{p,q}f$, where f is any primitive idempotent, prescribed by the Wedderburn-Artin Theorem, such that the decomposition (9) or (11) is valid, depending whether the algebra is simple or not. This part also reminds us that the left spinor ideals, while remaining left $C\ell_{p,q}$ modules, are right $\mathbb{K}$-modules. This is important when computing actual matrices in spinor representations (faithful and irreducible). Detailed computations of these representations in both simple and semisimple cases are shown in [2]. Furthermore, package `CLIFFORD` has a built-in database which displays matrices representing generators of $C\ell_{p,q}$, namely $\mathbf{e}_1, \ldots, \mathbf{e}_n$, $n = p + q$, for a certain choice of a primitive idempotent f. Then, the matrix representing any element $u \in C\ell_{p,q}$ can the be found using the fact that the maps γ shown on Parts (e) and (f), are algebra maps.

Finally, we should remark, that while for simple Clifford algebras the spinor minimal left ideal carries a **faithful** (and irreducible) representation, that is, $\ker\gamma = \{1\}$, in the case of semisimple algebras, each $\frac{1}{2}$ spinor space S and $\hat{S}$ carries an irreducible but not faithful representation. Only in the double spinor space $S\oplus\hat{S}$, one can realize the semisimple algebra faithfully. For all practical purposes, this means that each element u in a semisimple algebra must be represented by a pair of matrices, according to the isomorphism (11). In practice, the two matrices can then be considered as a single matrix, but over $\mathbb{K}\oplus\hat{\mathbb{K}}$ which is isomorphic to $\mathbb{R}\oplus\mathbb{R}$ or $\mathbb{H}\oplus\mathbb{H}$, depending whether $p-q = 1 \bmod 8$, or $p-q = 5 \bmod 8$. We have already remarked earlier that while $\mathbb{K}$ is a division ring, $\mathbb{K}\oplus\hat{\mathbb{K}}$ is not.

4. Conclusions

In this paper, the author has tried to show how the Structure Theorem on Clifford algebras $C\ell_{p,q}$ is related to the theory of semisimple rings, and, especially of left artinian rings. Detailed computations of spinor representations, which were distributed at the conference, came from [2].

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