DEPARTMENT OF MATHEMATICS

TECHNICAL REPORT

GRADED CLIFFORD ALGEBRAS AND GRADED SKEW CLIFFORD ALGEBRAS: A SURVEY OF THE ROLE OF THESE ALGEBRAS IN THE CLASSIFICATION OF ARTIN-SCHELTER REGULAR ALGEBRAS

P. VEERAPEN

October 2016

No. 2016-4

TENNESSEE TECHNOLOGICAL UNIVERSITY
Cookeville, TN 38505

Graded Clifford Algebras and Graded Skew Clifford Algebras: A Survey of the Role of these Algebras in the Classification of Artin-Schelter Regular Algebras

Padmini Veerapen

Abstract. This paper is a survey of work done on $\mathbb{N}$-graded Clifford algebras (GCAs) and $\mathbb{N}$-graded *skew* Clifford algebras (GSCAs) [VVW, SV, CaV, NVZ, VVe1, VVe2]. In particular, we discuss the hypotheses necessary for these algebras to be Artin Schelter-regular [AS, ATV1] and show how certain 'points' called, point modules, can be associated to them. We note that an AS-regular algebra can be viewed as a noncommutative analog of the polynomial ring. We begin our survey with a fundamental result in [VVW] that is essential to subsequent results discussed here: the connection between point modules and rank-two quadrics. Using, in part, this connection in [SV], the authors provide a method to construct GCAs with finitely many distinct isomorphism classes of point modules. After Cassidy and Vancliff go on to introduce a quantized analog of a GCA, called a graded *skew* Clifford algebra in [CaV], the authors in [NVZ] show that most Artin Schelter-regular algebras of global dimension three are either twists of graded skew Clifford algebras of global dimension three or Ore extensions of graded Clifford algebras of global dimension two. In [VVe1, VVe2], the authors go a step further and generalize the result of [VVW], between point modules and rank-two quadrics, by showing that point modules over GSCAs are determined by (noncommutative) quadrics of μ-rank at most two.

Mathematics Subject Classification (2010). Primary 16W50; Secondary 14A22.

Keywords. Clifford algebra, quadratic form, point module.

1. Introduction

In the 1980's, with the appearance of many noncommutative algebras from quantum physics and a desire to find a noncommutative algebraic geometry that would be as successful as commutative algebraic geometry had been for commutative algebra, M. Artin and W. Schelter [AS] introduced the notion of a regular algebra. A few years later Artin, Tate and Van den Bergh [AS, ATV1, ATV2] went on to classify the generic classes of regular algebras of global dimension three. The main idea behind this classification, introduced by Artin, Tate and Van den Bergh in [AS, ATV1, ATV2], involved using certain graded modules in place of geometric data, for example, point modules in place of certain points and line modules in place of certain lines. In particular, Artin, Tate, and Van den Bergh showed that such algebras could be associated to certain subschemes E (typically of dimension one) of $\mathbb{P}^2$ where points in the scheme E parametrize certain modules over these algebras called point modules. The technique involved the definition of a quantum analog of the projective plane $\mathbb{P}^2$. Since then progress has been made on the classification of regular algebras of global dimension four. However, this is still an open problem and as it stands now, *quadratic* regular algebras of global dimension four are still unclassified. This is where graded Clifford algebras (GCAs) and graded skew Clifford algebras (GSCAs) come into play.

In the papers we survey, we highlight how GCAs [Lb] and GSCAs [CaV] provide a trove of examples of regular algebras of global dimension three and four with the goal of informing the classification of quadratic algebras of global dimension four. We begin with a result from [VVW] where the authors prove a fundamental connection between GCAs and quadrics of rank at most two, stated precisely in Theorem 2.8. Using this result, we show how the authors in [SV] construct examples of regular GCAs with a finite number of point modules. In order to further inform the classification of quadratic Artin-Schelter regular (AS-regular) algebras of global dimension n, Cassidy and Vancliff in [CaV] introduced a quantized analog of a graded Clifford algebra called a graded *skew* Clifford algebra (GSCA).

Aiming to determine how useful GSCAs, the authors in [NVZ] prove that almost all quadratic regular algebras of global dimension three can be classified using GSCAs. Indeed, based on [AS] classification of regular algebras of global dimension three as being of types A, B, E, H, S_1, S_1', and S_2, Nafari, Vancliff, and Zhang showed that most quadratic AS-regular algebras of global dimension three are either twists of GSCAs of global dimension three or Ore extensions of graded Clifford algebras of global dimension two. In fact, only AS-regular algebras of type E and an open subset of those of type A could not be associated to GSCAs in that way.

In [VVe2], the authors generalize the connection between GCAs and quadrics of rank at most two from [VVW] to GSCAs by showing that point modules over GSCAs can be determined by noncommutative quadratic forms of μ-rank at most two [VVe1]. This notion of rank, called μ-rank, was defined on noncommutative quadratic forms, a generalization of traditional quadratic forms, in [VVe1] and this is used to determine point modules over GSCAs.

We conclude this survey paper with work being done in [ChV, TV] to construct two families of generic regular algebras of global dimension four where the main objective is to determine the line scheme of these families with the goal of ultimately classifying regular quadratic algebras of global dimension four.

In Section 2 of this paper, we present definitions of notions critical to the other sections and survey results about graded Clifford algebras (GCAs) from [VVW] and [SV]. In Section 3, we provide a review of results of [NVZ] and discuss how GSCAs of global dimension three are related to the types of AS-regular algebras classified by [ATV1, ATV2]. We also survey existing results from [VVe1] regarding the μ-rank of a noncommutative quadratic form as a generalization of the notion of rank of quadratic forms in the commutative setting and discuss results from [VVe2] between the μ-rank of noncommutative quadratic forms and the point modules over graded skew Clifford algebras. At the end of this paper, we mention in Remark 4.1 the work being done to relate 'classical' Clifford algebras as defined by Lounesto [L] to GSCAs.

2. Graded Clifford Algebras

Throughout the article, $\Bbbk$ denotes an algebraically closed field such that $\text{char}(\Bbbk) \neq 2$, and $M(n, \Bbbk)$ denotes the vector space of $n \times n$ matrices with entries in $\Bbbk$. For a graded $\Bbbk$-algebra B, the span of the homogeneous elements in B of degree i will be denoted B_i, and the notation $T(V)$ will denote the tensor algebra on the vector space V. If C is any ring or vector space, then $C^\times$ will denote the nonzero elements in C. We use R to denote the polynomial ring on degree-one generators $x_1, \ldots, x_n$.

In this section, we survey results on graded Clifford algebras (GCAs). In particular, we discuss methods used to determine point modules over GCAs. We begin with the definition of a GCA and provide other useful definitions.

Definition 2.1. [Lb, §4] Let $M_1, \ldots, M_n \in M_n(\Bbbk)$ be symmetric matrices. A graded Clifford algebra is the $\Bbbk$-algebra C on degree-one generators $x_1, \ldots, x_n$ and on degree-two generators $y_1, \ldots, y_n$ with the following defining relations:

(a) $x_i x_j + x_j x_i = \sum_{k=1}^{n} (M_k)_{ij} y_k$ for all $i, j = 1, \ldots, n$;

(b) y_k central for all $k = 1, \ldots, n$.

Remark 2.2. (i) Suppose B is a quadratic regular algebra where $V = B_1$ and $W \subset V \otimes_{\Bbbk} V$ be the span of the defining relations of B and let $\mathcal{V}(W)$ be the zero locus in $\mathbb{P}(V^*) \times \mathbb{P}(V^*)$ of elements of W. The Koszul dual [ST], B^*, of B is the quadratic algebra given by the quotient of the tensor algebra on V^* by the ideal generated by $W^\perp$ where $W^\perp = \{f \in V^* \otimes_{\Bbbk} V^* : f(w) = 0 \text{ for } w \in W\}$.

(ii) With $\mathcal{V}(W)$ as in part (i), by [ATV1] $\mathcal{V}(W)$ is the graph of an automorphism τ of a scheme $E \subset \mathbb{P}^2$.

Example. Suppose $M_1 = \left[\begin{smallmatrix} 2 & \lambda \\ \lambda & 0 \end{smallmatrix}\right]$ and $M_2 = \left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right]$, where $\lambda \in \Bbbk$. Let C be the graded $\Bbbk$-algebra on degree-one generators x_1, x_2 and on degree-two generators y_1, y_2. Using the relations in Definition 2.1(i), we obtain $2x_1^2 = 2y_1$, $x_2^2 = y_2$, and $x_1x_2 + x_2x_1 = \lambda y_1 = \lambda x_1^2$. Thus, $\frac{\Bbbk\langle x_1, x_2\rangle}{\langle x_1x_2 + x_2x_1 - \lambda x_1^2\rangle}$ maps onto C.

Definition 2.3. [AS] Let $A = \oplus_{i=0}^{\infty} A_i$ be a finitely generated, $\mathbb{N}$-graded, connected $\Bbbk$-algebra. The algebra A will be called regular (or AS-regular) if it satisfies the following properties:

(a) A has finite global dimension d,
(b) A has polynomial growth, and
(c) A is Gorenstein.

Remark 2.4. An equivalent condition to condition (b) is that A has finite Gelfand-Kirillov dimension (see [KL]). Condition (c) is the main reason why an AS-regular algebra is viewed as a noncommutative analogue to the polynomial ring inasmuch as it replaces the symmetry condition of commutativity, as shown in the example below. Henceforth, an AS-regular algebra will be called a regular algebra.

Example. If the $\Bbbk$-algebra $A = \Bbbk[x, y]$ where $xy = qyx$ and $q \in \Bbbk^{\times}$, then A is Gorenstein. Consider the minimal projective resolution of ${}_A\Bbbk$.

$$0 \longrightarrow {}_AA \xrightarrow{\ h\ } {}_AA^2 \xrightarrow{\ f\ } {}_AA \longrightarrow {}_A\Bbbk \longrightarrow 0 \qquad (*),$$

where $h(a) = a\begin{bmatrix} -qy \\ x \end{bmatrix}$ and $f(b, c) = \begin{bmatrix} b & c \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$ for $a, b, c \in A$. This minimal projective resolution is a short exact sequence for A. If we now dualize $(*)$ by applying the functor $GHom(.,_A A)$, we obtain the following minimal resolution

$$0 \longrightarrow A_A \xrightarrow{\ f^*\ } A_A^2 \xrightarrow{\ h^*\ } A_A \longrightarrow \Bbbk_A \longrightarrow 0,$$

where $f^*(c) = \begin{bmatrix} x \\ y \end{bmatrix} c$ and $h^* = \begin{bmatrix} -qy & x \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix}$ for $a, b, c \in A$. Note that f^* and h^* are now left multiplication by the same matrices as in $(*)$.

Remark 2.5. As discussed in [VVW, CaV], we may associate geometry to a GCA by considering the quadratic form associated to the symmetric matrix M_k. We do so by associating a quadratic form $q_k \in \Bbbk[z_1, \ldots, z_n]$ to each M_k via $q_k = z^T M_k z$ with $z^T = [z_1 \ldots z_n]$. In particular, viewing the basis $\{z_1, \ldots, z_n\}$ as the dual basis to $\{x_1, \ldots, x_n\}$, the (commutative) polynomial algebra $\Bbbk[z_1, \ldots, z_n]$, R, is the Koszul dual of the quadratic algebra $B/\langle y_1, \ldots, y_n\rangle$. Moreover, the zero locus $\mathcal{V}(q_k)$ of each q_k is a *quadric* in $\mathbb{P}(C_1^*)$. A *base point* of the quadric system $\mathcal{V}(q_1), \ldots, \mathcal{V}(q_n)$ is a point in $\bigcap_{k=1}^n \mathcal{V}(q_k)$. If no such point exists, then the quadric system is said to be base point free.

Results in [AL, Lb] relates GCAs to regular algebras using the geometry we discuss above.

Proposition 2.6. [AL, Lb] *The graded Clifford algebra C is quadratic, Auslander-regular of global dimension n and satisfies the Cohen-Macaulay property with Hilbert series $1/(1-t)^n$ if and only if the associated quadric system is base point free; in this case, C is Artin-Schelter regular and is a noetherian domain.*

Example. Let M_1 and M_2 be as in the previous example. Since the quadratic forms q_1 and q_2 associated to M_1 and M_2, respectively, are base point free, then the $\Bbbk$-algebra $\frac{\Bbbk\langle x_1, x_2\rangle}{\langle x_1x_2+x_2x_1-\lambda x_1^2\rangle}$ is isomorphic to the Clifford algebra, C, by Proposition 2.6.

2.1. Graded Clifford Algebras and Quadrics of Rank Two

We now want to determine certain modules called, point modules, over regular graded Clifford algebras. We begin with a definition and examine a result in [VVW] that relates point modules to a graded Clifford algebra. It is helpful to note that in one of their seminal papers, [ATV1] prove that, under certain conditions, point modules are determined by a scheme. This scheme represents the functor of point modules and was called in [VVr], the *point scheme*. The point scheme of regular algebras will be essential to the classification of such algebras in this article.

Definition 2.7. [ATV1] A right (respectively, left) point module over a graded algebra, B, is a cyclic graded right (respectively, left) B-module $M = \bigoplus_{i\geq 0} M_i$ such that $M = M_0B$ or (respectively, BM_0) and $\dim_{\Bbbk}(M_i) = 1$ for all i.

Theorem 2.8. [VVW, Theorem 1.7] *Let C denote a GCA determined by symmetric matrices, $N_1, \dots, N_n \in M(n, \Bbbk)$ and let $\mathcal{Q}$ be the corresponding (commutative) quadric system in $\mathbb{P}^{n-1}$. If $\mathcal{Q}$ is base point free, then the number of isomorphism classes of left (respectively, right) point modules over C is equal to $2r_2 + r_1 \in \mathbb{N} \cup \{0, \infty\}$ where r_j denotes the number of matrices in $\mathbb{P}\left(\sum_{k=1}^{n} \Bbbk N_k\right)$ that have rank j. If the number of left (respectively, right) point modules is finite, then $r_1 \in \{0, 1\}$.*

This result, Theorem 2.8, is a fundamental one in that it allows us to control the number of point modules over regular GCAs using the the quadrics of rank at most two in the Koszul dual of the GCA.

2.2. Graded Clifford Algebras with Finitely Many Points

We will now discuss how to construct certain regular graded Clifford algebras of global dimension four with a finite number of point modules by surveying results in [SV]. The authors in [SV] use Theorem 2.8 to construct regular GCAs of global dimension four with a finite number of points in their scheme. Indeed, if a GCA has at least two and at most finitely many distinct isomorphism classes of point modules, then [SV] show that the number of intersection points of two planar cubic divisors determines the number of isomorphism classes of point modules over such GCAs. The two planar cubic divisors, in turn, parametrize quadrics of rank at most two. This is where the result that $r_1 + 2r_2$ where r_i refers to the number of quadrics of rank i from Theorem 2.8 comes into play. For this subsection, we will call a linear system of quadrics $\mathfrak{q}$, such that $\mathbb{P}(\mathfrak{q}) \cong \mathbb{P}^2$, a net of quadrics and a linear system of quadrics $\mathfrak{Q}$, such that $\mathbb{P}(\mathfrak{Q}) \cong \mathbb{P}^3$, a web of quadrics.

Using Proposition 2.6, the authors determine whether a GCA of global dimension four is regular. They do so by ensuring that the quadric system associated to the GCA is base point free. The method used to construct regular GCAs with a finite number of points in their point scheme, involves the use of two planar cubic divisors. Suppose $\mathfrak{Q} = \{q_1, q_2, q_3, q_4\}$ where $q_1 = \mathcal{V}(a_1 a_2)$ is a web of quadrics in $\mathbb{P}^3$ corresponding to a regular GCA of global dimension four. Recall that from the discussion in Remark 2.5, we can associate a linear system of quadrics to a GCA via its Koszul dual. We consider the quadratic form $Q = a_1 a_2 \in R_2$ and the net $\mathfrak{q} = \Bbbk q_2 \oplus \Bbbk q_3 \oplus \Bbbk q_4$ such that $Q \notin \mathfrak{q}$. We view $\mathfrak{q}_i$ as the isomorphic image of the net $\mathfrak{q}$ determined by the map $R_2 \to R_2/R_1 a_i$ and let the schemes F_i be the family of quadrics in $\mathbb{P}(\mathfrak{q})$ that meet $\mathcal{V}(a_i)$ in a degenerate conic. In fact, F_i parametrizes the quadrics in $\mathbb{P}(\mathfrak{q}_i)$ of rank two and $\dim(F_i) = 1$ for $i = 1, 2$. The following result states precisely how the intersection points of the cubic divisors determine the number of points in the point scheme of the GCA:

Theorem 2.9. [SV, Theorem 2.6] *Let A be a graded Clifford algebra of global dimension four and let $\mathfrak{Q}$ denote the corresponding quadric system that is base-point free. Suppose $\mathfrak{Q} \oplus \mathfrak{q}$, where $\mathfrak{q}$ is a net of quadrics in $\mathfrak{Q}$ and $Q = \mathcal{V}(ab) \notin \mathfrak{q}$ such that $a, b \in R_1$ are linearly independent. Write F_1 and F_2 for the two planar cubic divisors in $\mathbb{P}(\mathfrak{q})$ that parametrize the quadrics in $\mathbb{P}(\mathfrak{q})$ that meet the two distinct planes of Q in a degenerate conic such that $|F_1 \cap F_2| < \infty$ (that is, $|F_1 \cap F_2| \leq 9$). Let m be the number of distinct points in $F_1 \cap F_2$, and let r_i denote the number of distinct quadrics in $\mathbb{P}(\mathfrak{Q})$ of rank i, and suppose $r_1 \leq 1$. We have*

(a) $r_2 < \infty$;
(b) A has at most finitely many isomorphism classes of point modules;
(c) $m - r_1 \leq r_2 \leq 2(m - r_1) + 1$.

The following result will be helpful in the example below:

Lemma 2.10. [SV, Lemma 2.3] *If Q and Q' are distinct quadrics in $\mathbb{P}^3$, then either $\mathbb{P}(\Bbbk Q \oplus \Bbbk Q')$ consists of quadrics of rank at most two or $\mathbb{P}(\Bbbk Q \oplus \Bbbk Q')$ contains at most three quadrics of rank at most two. Moreover, if $\mathbb{P}(\Bbbk Q \oplus \Bbbk Q')$ consists of quadrics of rank at most two, then if $Q = \mathcal{V}(ab)$, where $a, b \in R_1$ are linearly independent, then $Q' = \mathcal{V}(f)$ where $f \in R_1 a$ or $f \in R_1 b$ or $f \in \Bbbk^\times a^2 + \Bbbk ab + \Bbbk^\times b^2$.*

Example. [SV, Example 5.1] The goal in this example is to construct a regular GCA with exactly twenty distinct points in its point scheme. First, for the GCA to be regular, one needs a base-point free web $\mathfrak{Q} \in \mathbb{P}^3$ of quadrics such that $\mathbb{P}(\mathfrak{Q})$ contains exactly ten rank-two quadrics. Recall that the ongoing assumption is that the GCA has more than one point in its point scheme so, using the fact that any regular GCA of global dimension four has $r_1 + 2r_2$ point modules, this means that $\mathfrak{Q}$ will contain at least one quadric of rank two.

Suppose $a, b, c, d \in R_1$ are linearly independent and suppose $\mathfrak{q} = \Bbbk(a^2 - b^2) \oplus \Bbbk(a^2 - c^2) \oplus \Bbbk(a^2 - d^2)$. We see that $\mathbb{P}(\mathfrak{q})$ contains six rank-two quadrics and none of rank one. We set $Q = a^2 - e^2$, for some $e \in S_1$, and define $\mathfrak{Q} = \Bbbk Q \oplus \mathfrak{q}$. If e is taken to equal $a + b + c + d$, then $\mathfrak{Q}$ is a base-point free web of quadrics (if $\mathrm{char}(\Bbbk) \neq 3$ or 5) such that $\mathbb{P}(\mathfrak{Q})$ contains exactly ten rank-two quadrics. One can then set $a = \pm b$ and obtain two cubic divisors with their six intersection points. The number of rank-two quadrics are obtained by linearly combining the quadrics corresponding to the intersection points with $a^2 - b^2$.

3. Graded Skew Clifford Algebras

In [CaV], Cassidy and Vancliff introduced a quantized analog of a graded Clifford algebra, known as a graded *skew* Clifford algebra (GSCAs). The introduction of this quantized analog required several notions to be generalized: μ-symmetric matrices, noncommutative quadratic forms, noncommutative quadric system, base points. We discuss these generalizations and survey papers that generalize results from the previous section, regarding graded Clifford algebras, to graded *skew* Clifford algebras. In particular, in [NVZ], we show how GSCAs come into play with regard to most quadratic regular algebras of global dimension three. Moreover, in [VVe1, VVe2], we discuss a generalization to the notion of rank of a noncommutative quadratic form and provide a generalization to Theorem 2.8 in [VVW].

3.1. Graded Skew Clifford Algebras

We write S as in [CaV] for the quadratic $\Bbbk$-algebra on degree-one generators $z_1, \ldots, z_n$ and defining relations $z_j z_i = \mu_{ij} z_i z_j$ for all i, $j = 1, \ldots, n$, where $\mu_{ii} = 1$ for all i. That is,

$$S = \frac{T(V)}{\langle z_j z_i - \mu_{ij} z_i z_j : i, j = 1, \cdots, n \rangle},$$

where $T(V)$ is the tensor algebra on $V = \mathrm{span}\{z_1, \cdots, z_n\}$. We set $U \subset S_1 \otimes_{\Bbbk} S_1$ to be the span of defining relations of S and write $z = (z_1, \ldots, z_n)^T$.

Definition 3.1. [CaV, §1.2]

(a) With μ and S as above, a quadratic form Q is any element of S_2. Identifying $\mathbb{P}(S_1^*)$ with $\mathbb{P}^{n-1}$, we note that the subscheme of the zero locus $Z \subset \mathbb{P}^{n-1} \times \mathbb{P}^{n-1}$ of U consisting of all points in Z on which a quadratic form q vanishes is called the quadric determined by q.

(b) A matrix $M \in M(n, \Bbbk)$ is called μ-symmetric if $M_{ij} = \mu_{ij} M_{ji}$ for all i, $j = 1, \ldots, n$.

We write $M^{\mu}(n, \Bbbk)$ for the set of μ-symmetric matrices in $M(n, \Bbbk)$ and note that if $\mu_{ij} = 1$ for all i, j, then $M^{\mu}(n, \Bbbk)$ is the set of all symmetric matrices.

Remark 3.2. For $\{i,\ j\} \subset \{1, \ldots, n\}$, let $\mu_{ij} \in \Bbbk^{\times}$ such that $\mu_{ij}\mu_{ji} = 1$ for all $i \neq j$ and we write $\mu = (\mu_{ij}) \in M(n,\ \Bbbk)$.

Definition 3.3. [CaV] A *graded skew Clifford algebra* $A = A(\mu, M_1, \ldots, M_n)$ associated to μ and $M_1, \ldots, M_n$ is a graded $\Bbbk$-algebra on degree-one generators $x_1, \ldots, x_n$ and on degree-two generators $y_1, \ldots, y_n$ with defining relations given by:

(a) $x_i x_j + \mu_{ij} x_j x_i = \sum_{k=1}^{n} (M_k)_{ij} y_k$ for all $i, j = 1, \ldots, n$, and

(b) the existence of a normalizing sequence $\{y_1', \ldots, y_n'\}$ that spans $\Bbbk y_1 + \cdots + \Bbbk y_n$.

Remark 3.4. If A is a GSCA, then [CaV, Lemma 1.13] implies that $y_i \in (A_1)^2$ for all $i = 1, \ldots, n$ if and only if $M_1, \ldots, M_n$ are linearly independent. Henceforth, we assume that $M_1, \ldots, M_n$ are linearly independent.

Example. Consider the following four μ-symmetric matrices: $M_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \mu_{21} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$, $M_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 \\ \mu_{31} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $M_3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ \mu_{41} & 0 & 0 & 0 \end{bmatrix}$, $M_4 = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & \mu_{32} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

Using Definition 3.3(a), we obtain the following relations:

$$x_1x_2 + \mu_{12}x_2x_1 = x_4^2, \quad x_1x_4 + \mu_{14}x_4x_1 = x_2^2, \qquad x_2x_4 = -\mu_{24}x_4x_2,$$

$$x_1x_3 + \mu_{13}x_3x_1 = x_3^2, \qquad x_2x_3 + \mu_{23}x_3x_2 = x_1^2, \quad x_3x_4 = -\mu_{34}x_4x_3.$$

We know that, A', the $\Bbbk$-algebra on x_1, x_2, x_3, x_4 factored out by the ideal generated by the above relations, is mapped onto a GSCA. However, since x_1^2, x_2^2, x_3^2, and x_4^2 are normal in A', then A' is in fact a GSCA.

Theorem 3.5. [CaV, Theorem 4.2] *A graded skew Clifford algebra* $A = A(\mu, M_1, \ldots, M_k)$ *is a quadratic Auslander-regular algebra of global dimension n that satisfies the Cohen-Macaulay property with Hilbert series* $1/(1-t)^n$ *if and only if the quadric system association to* $\{q_1, \ldots, q_n\}$ *is normalizing and base point free; in this case, A is an Artin-Schelter regular* $\mathbb{N}$*-graded* $\Bbbk$*-algebra of global dimension n, a noetherian domain and unique up to isomorphism.*

Definition 3.6.

(a) [CaV] The span of quadratic forms $Q_1, \ldots, Q_m \in S_2$ will be called the *quadric system* associated to $Q_1, \ldots, Q_m$. If a quadric system is given by a normalizing sequence in S, then it is called a *normalizing quadric system*.

(b) [CaV] We define a *left base point* of a quadric system $\mathfrak{Q} \subset S_2$ to be any left base-point module over $S/\langle \mathfrak{Q} \rangle$; that is, to be any 1-critical graded left module over $S/\langle\mathfrak{Q}\rangle$ that is generated by its homogeneous degree-zero elements and which has Hilbert series $H(t) = c/(1-t)$, for some $c \in \mathbb{N}$. We say a quadric system is *left base-point free* if it has no left base points. Similarly, for right base point, etc.

(c) [ATV2] In (b), if $c = 1$, then a left base-point module is cyclic and is a (left) point module.

(d) [ATV1] In (b), if $c \geq 2$, then a left base-point module is a (left) fat point module.

Remark 3.7. In [CaV, Corollary 11], a normalizing quadric system $\mathfrak{Q}$ is right base-point free if and only if $\dim_{\mathbb{k}}(S/\langle\mathfrak{Q}\rangle) < \infty$. Thus, a normalizing quadric system $\mathfrak{Q}$ is right base-point free if and only if it is left base-point free.

3.2. Classification of quadratic regular algebras of global dimension three via GSCAs

We now survey, Nafari, Vancliff, and Zhang's results [NVZ] on the classification of most quadratic regular algebras of global dimension three as graded skew Clifford algebras. The authors consider all possible 'types' of quadratic regular algebras of global dimension three and show they can be viewed as regular quadratic graded skew Clifford algebras. The analysis is split into two main cases: quadratic regular algebras of global dimension three with reducible or non-reduced point scheme (point scheme contains a line) and quadratic regular algebras of global dimension three with irreducible or reduced point scheme. Recall that the point scheme of a quadratic regular algebra represents the functor of point modules. By [ATV1], the point scheme of a quadratic regular algebra of global dimension three is a cubic divisor in $\mathbb{P}^2$ or is all of $\mathbb{P}^2$. The next result relates algebras whose point scheme contains a line, that is whose point scheme is reducible or non-reduced, to GSCAs.

Theorem 3.8. [NVZ, Theorem 1.10] *Suppose A is a quadratic AS-regular algebra of global dimension three, suppose $C \in \mathbb{P}^2$ is its associated point scheme and suppose char$(\mathbb{k}) \neq 2$. If the point scheme of A is reducible or non-reduced, then either*

(a) A is a twist, by an automorphism, of a GSCA, or
(b) A is a twist, by a twisting system, of an Ore extension of a regular GSCA of global dimension two.

This result provides us with several examples of quadratic AS-regular algebras of global dimension three via GSCAs. The following example [NVZ, Example 1.9] is that of a regular GSCA of global dimension three with a reducible or non-reduced point scheme, that is, a point scheme that contains a line.

Example. Suppose that char$(\mathbb{k}) \neq 2$ and that μ_{ij}'s are as in Remark 3.2. Let $M_1 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $M_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, and $M_3 = \begin{bmatrix} 0 & 1 & 0 \\ \mu_{21} & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ and fix $\mu_{13}\mu_{23} = 1$ and $\mu_{13} + \mu_{12}\mu_{23} \neq 0$. Using Theorem 3.5, we can show that the μ-symmetric matrices M_1, M_2 and M_3 yield a regular GSCA, A, of global dimension three on generators x_1, x_2, x_3 with defining relations

$$x_1x_2 + \mu_{12}x_2x_1 = 0.$$
$$x_1x_3 + \mu_{13}x_3x_1 = 0.$$
$$x_2x_3 + \mu_{23}x_3x_2 = x_3^2.$$

The point scheme of A is isomorphic to a subscheme $\mathcal{P}$ of $\mathbb{P}^2$. The subscheme $\mathcal{P}$ can computed by considering all the points $((a_1, a_2, a_3), (b_1, b_2, b_3))$ in $\mathbb{P}^2 \times \mathbb{P}^2$ such that

$$(x_1x_2 + \mu_{12}x_2x_1)((a_1, a_2, a_3), (b_1, b_2, b_3)) = 0, \text{ and}$$
$$(x_1x_3 + \mu_{13}x_3x_1)((a_1, a_2, a_3), (b_1, b_2, b_3)) = 0, \text{ and}$$
$$(x_2x_3 + \mu_{23}x_3x_2 - x_3^2)((a_1, a_2, a_3), (b_1, b_2, b_3)) = 0.$$

That is, solve

$$\begin{bmatrix} \mu_{12}a_2 & a_1 & -a_3 \\ \mu_{13}a_3 & 0 & a_1 \\ 0 & \mu_{23}a_3 & a_2 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

To find solutions in $\mathbb{P}^2 \times \mathbb{P}^2$, we may assume that the determinant of the above 3×3 matrix is zero. This yields $((\mu_{13} + \mu_{12}\mu_{23})a_1a_2 + a_3^2)a_3 = 0$. That is, $\mathcal{P} = \mathcal{V}(((\mu_{13}+\mu_{12}\mu_{23})x_1x_2+x_3^2)x_3)$, which is the union of a nondegenerate conic and a line.

For the rest of this section, the authors consider regular algebras of global dimension three with a point scheme given by a cubic divisor C that is reduced and irreducible. Such point schemes do not contain lines and thus, the cubic divisor has at most one singular point. Moreover, x, y and z are used for homogeneous degree-one linearly independent (commutative) coordinates in $\mathbb{P}^2$. It should be noted that algebras with nodal cubic and cuspidal cubic curves as point schemes are not discussed in [ATV1] since they are not generic.

In [NVZ, Lemma 2.1], an irreducible cubic divisor, C in $\mathbb{P}^2$, with a unique singular point is given by $\mathcal{V}(f)$ where either (a) $f = x^3 + y^3 + xyz$, or (b) $f = y^3 + x^2z$, or (c) $f = y^3 + x^2z + xy^2$ and if $\text{char}(\Bbbk) \neq 3$, then f is given by (a) or (b). If C is given by (a), then C is called a nodal cubic curve and called a cuspidal cubic curve, otherwise. In both cases where the algebras' point scheme is given by either a nodal cubic curve or cuspidal cubic curve, the authors show that the algebras are in fact isomorphic to graded skew Clifford algebras or Ore extensions of regular GSCAs.

In [AS] regular algebras of global dimension three with point scheme an elliptic curve are classified into four types: A, B, E and H. These types along with their relations are summarized in the table below. The reader should note that some members of each type might not have an elliptic curve as their point scheme, but a generic member does. In [NVZ], the authors show that algebras of types H, B, and some of type A can be related to graded *skew* Clifford algebras and graded Clifford algebras. Algebras of type E and an open subset of those of type A were not tied to GSCAs.

Type	Defining Relations	Conditions
H	$y^2 = x^2$, $zy = -iyz$, $yx - xy = iz^2$, where i is a primitive fourth root of unity.	$\text{char}(\Bbbk) \neq 2$
B	$xy + yx = z^2 - y^2$, $xy + yx = az^2 - x^2$, $zx - xz = a(yz - zy)$, $a \in \Bbbk$,	$a(a-1) \neq 0$, $\text{char}(\Bbbk) \notin \{2, 3\}$
A	$axy + byx + cz^2 = 0$, $ayz + bzy + cx^2 = 0$, $azx + bxz + cy^2 = 0$, $a, b, c \in \Bbbk$.	$abc \neq 0$, $\text{char}(\Bbbk) \neq 3$ $(3abc)^3 \neq (a^3 + b^3 + c^3)^3$, $a^3 = b^3 \neq c^3$, $b^3 = c^3 \neq a^3$, $a^3 = c^3 \neq b^3, a^3 \neq b^3 \neq c^3$
E	$\gamma zx + xz = -\gamma^5 y^2$, $yx + \gamma^4 xy = -\gamma^2 z^2$, $zy + \gamma^7 yz = -\gamma^8 x^2$, where γ is a primitive ninth root of unity.	

The algebras of type H, B, and A will be denoted by H, B, and A', respectively.

Lemma 3.9. [NVZ]

(a) If char($\Bbbk$) $\neq 2$, then the algebra H is a regular graded skew Clifford algebra.
(b) If char($\Bbbk$) $\neq 2$, then the algebra B is regular if and only if $a^2 - a + 1 \neq 0$; in this case, B is a graded skew Clifford algebra and a twist of a Clifford algebra by an automorphism.
(c) Suppose char($\Bbbk$) $\neq 2$. If $a^3 = b^3 \neq c^3$, then A' is a regular graded skew Clifford algebra and a twist of a graded Clifford algebra by an automorphism. If $b^3 = c^3 \neq a^3$ or if $a^3 = c^3 \neq b^3$, then A' is a twist of a regular graded skew Clifford algebra by an automorphism.

Proof. (Sketch of proof) The main idea for the proof to parts (a), (b), and (c) is to use Remark 2.2 parts (i) and (ii) to compute the Koszul dual of H, B, and A' respectively, and apply Theorem 4.2 [CaV] to a certain normalizing sequence with empty zero locus in $\mathbb{P}^2 \times \mathbb{P}^2$. The proof to part (c) uses a similar argument for each of the three cases except that the three cases reduce to one case when A' is twisted. □

Remark 3.10. We note that that a GCA (discussed in §2) is a finite module over some commutative subalgebra C, while a GSCA is a finite module over a (likely noncommutative) subalgebra R. In [NV], the authors show that if a regular GSCA is a twist by an automorphism of a GCA, then the subalgebra

R is a skew polynomial ring and is a twist of the commutative subalgebra C by an automorphism.

3.3. Point Modules over GSCAs

In this section, we discuss results [VVe1, VVe2] that connect point modules to GSCAs. We see in Theorem 3.17 using the definition of μ-rank from Section 2 that the quadratic forms of μ-rank at most two control the point modules over GSCAs. This result generalizes Theorem 2.8 that related (commutative) quadratic forms of rank at most two to point modules over GCAs. We begin this section with a generalization of the notion of rank, called μ-rank, for noncommutative quadratic forms.

As in previous sections, for $\{i,\ j\} \subset \{1,\dots,n\}$, let $\mu_{ij} \in \Bbbk^\times$ such that $\mu_{ij}\mu_{ji} = 1$ for all $i \neq j$ and we write $\mu = (\mu_{ij}) \in M(n,\ \Bbbk)$ and

$$S = \frac{T(V)}{\langle z_j z_i - \mu_{ij} z_i z_j : i, j = 1, \cdots, n\rangle},$$

where $T(V)$ is the tensor algebra on $V = \operatorname{span}\{z_1, \cdots, z_n\}$ and we write $z = (z_1, \dots, z_n)^T$.

Definition 3.11. [VVe1] Let $M = \begin{bmatrix} a & d & e \\ \mu_{21}d & b & f \\ \mu_{31}e & \mu_{32}f & c \end{bmatrix} \in M^\mu(3, \Bbbk)$ and, for $1 \leq i \leq 8$, define the functions $D_i : M^\mu(3, \Bbbk) \to \Bbbk$ by

$$D_1(M) = 4d^2 - (1+\mu_{12})^2 ab, \quad D_4(M) = 2(1+\mu_{23})de - (1+\mu_{12})(1+\mu_{13})af,$$
$$D_2(M) = 4e^2 - (1+\mu_{13})^2 ac, \quad D_5(M) = 2(1+\mu_{12})ef - (1+\mu_{13})(1+\mu_{23})cd,$$
$$D_3(M) = 4f^2 - (1+\mu_{23})^2 bc, \quad D_6(M) = 2(1+\mu_{13})df - (1+\mu_{12})(1+\mu_{23})be,$$

$$D_7(M) = (\mu_{23}cd^2 - 2def + be^2)(\mu_{13}\mu_{21}cd^2 - 2def + \mu_{12}\mu_{23}\mu_{31}be^2),$$

$$D_8(M) = \mu_{21}(d+X)(e-Y) + \mu_{23}\mu_{31}(d-X)(e+Y) - 2af,$$

where $X^2 = d^2 - \mu_{12}ab$ and $Y^2 = e^2 - \mu_{13}ac$. We call $D_1, \dots, D_6$ the 2×2 μ-minors of M and D_7 and D_8 will play a role analogous to the μ-determinants of M.

Theorem 3.12. [VVe1] *Let $Q = az_1^2 + bz_2^2 + cz_3^2 + 2dz_1z_2 + 2ez_1z_3 + 2fz_2z_3 \in S_2$, where $a, \dots, f \in \Bbbk$, and let $M \in M^\mu(3,\ \Bbbk)$ be the μ-symmetric matrix associated to Q.*

(a) *There exists $L \in S_1$ such that $Q = L^2$ if and only if $D_i(M) = 0$ for all $i = 1, \dots, 6$.*

(b) (i) *If $a = 0$, then there exists $L_1,\ L_2 \in S_1$ such that $Q = L_1L_2$ if and only if $D_7(M) = 0$;*

(ii) *if $a \neq 0$, then there exists $L_1,\ L_2 \in S_1$ such that $Q = L_1L_2$ if and only if $D_8(M) = 0$ for some X and Y satisfying $X^2 = d^2 - \mu_{12}ab$ and $Y^2 = e^2 - \mu_{13}ac$.*

Remark 3.13. Let $\tau : \mathbb{P}(M^\mu(n,\Bbbk)) \to \mathbb{P}(S_2)$ be defined by $\tau(M) = z^T M z$. Hereafter, we fix $M_1, \dots, M_n \in M^\mu(n,\Bbbk)$. For each $k = 1, \dots, n$, we fix representatives $q_k = \tau(M_k)$. Moreover, if M is a μ-symmetric matrix in $\mathbb{P}(M^\mu(n,\Bbbk))$ and if $\mu\text{-rank}(T(M)) \le 2$, then we define $\mu\text{-rank}(M)$ to be μ-rank of $\tau(M)$.

Remark 3.14. If A is a GSCA, then [CaV, Lemma 1.13] implies that $y_i \in (A_1)^2$ for all $i = 1, \dots, n$ if and only if $M_1, \dots, M_n$ are linearly independent. Henceforth, we assume that $M_1, \dots, M_n$ are linearly independent.

Example. Consider the following four μ-symmetric matrices: $M_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \mu_{21} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$, $M_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 \\ \mu_{31} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $M_3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ \mu_{41} & 0 & 0 & 0 \end{bmatrix}$, $M_4 = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & \mu_{32} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

Using Definition 3.3(a), we obtain the following relations:

$$x_1x_2 + \mu_{12}x_2x_1 = x_4^2, \quad x_1x_4 + \mu_{14}x_4x_1 = x_2^2, \qquad x_2x_4 = -\mu_{24}x_4x_2,$$

$$x_1x_3 + \mu_{13}x_3x_1 = x_3^2, \qquad x_2x_3 + \mu_{23}x_3x_2 = x_1^2, \quad x_3x_4 = -\mu_{34}x_4x_3.$$

We know that, A', the $\Bbbk$-algebra on x_1, x_2, x_3, x_4 factored out by the ideal generated by the above relations, is mapped onto a GSCA. However, since x_1^2, x_2^2, x_3^2, and x_4^2 are normal in A', then A' is in fact a GSCA.

Theorem 3.15. [CaV] *For all $k = 1, \dots, n$, let M_k and q_k be as in Remark* 3.13*. A graded skew Clifford algebra $A = A(\mu, M_1, \dots, M_n)$ is a quadratic, Auslander-regular algebra of global dimension n that satisfies the Cohen-Macaulay property with Hilbert series $1/(1-t)^n$ if and only if the quadric system associated to $\{q_1, \dots, q_n\}$ is normalizing and base-point free; in this case, A is a noetherian Artin-Schelter regular domain and is unique up to isomorphism.*

Remark 3.16. (a) Hereafter, we assume that the quadric system associated to $q_1, \dots, q_n$ is normalizing and base-point free. Theorem 3.15 allows us to write $A = T(V)/\langle W\rangle$, where $V = S_1^*$ and $W \subset T(V)_2$. We write the Koszul dual [ST] of A as $T(S_1)/\langle W^\perp\rangle = S/\langle q_1, \dots, q_n\rangle$. We note that $\{x_1, \dots, x_n\}$ is the dual basis in V to the basis $\{z_1, \dots, z_n\}$ of S_1 and we write $\sum_{i,j} \alpha_{ijm}(x_ix_j + \mu_{ij}x_jx_i) = 0$ for the defining relations of A, where $\alpha_{ijm} \in \Bbbk$ for all i, j, m, and $1 \le m \le n(n-1)/2$.

(b) By [CaV, Lemma 5.1], there is a one-to-one correspondence between the set of pure tensors in $\mathbb{P}(W^\perp)$, that is, $\{a \otimes b \in \mathbb{P}(W^\perp) : a, b \in S_1\}$ and the zero locus Γ of W given by $\Gamma = \{(a, b) \in \mathbb{P}(S_1) \times \mathbb{P}(S_1) : w(a, b) = 0 \text{ for all } w \in W\}$.

Example. We consider the quadric system $\mathfrak{Q}$ given by $q_1 = z_1z_2, q_2 = z_3^2, q_3 = z_1^2 - z_2z_4$, and $q_4 = z_2^2 + z_4^2 - z_2z_3$ in [CaV, §5.3]. We observe that $\mathfrak{Q}$ is a normalizing and base-point free quadric system if and only if $\mu_{34}^2 = \mu_{23} = 1$ and $\mu_{34} = \mu_{24} = (\mu_{14})^2 = (\mu_{13})^2$. The quadric system is base-point free since the ideal I it generates contains $z_1^3, z_2^5, z_3^2, z_4^5$. Thus, $\dim_{\Bbbk}(S/I) < \infty$ and [CaV, Corollary 11] implies that $\mathfrak{Q}$ is a base-point free quadric system.

In order to relate point modules to GSCAs, the authors consider the quadratic forms of μ-rank at most two that are in the Koszul dual (see Definition 2.2) of the GSCA. First, recall that the authors are considering regular GSCAs and so, apply Theorem 3.15 to obtain a base point free normalizing quadric system. A point (a, b) is associated to a μ-symmetric matrix via the map Φ from $\mathbb{P}^{n-1} \times \mathbb{P}^{n-1}$ to $\mathbb{P}(M^\mu(n, \Bbbk))$ and the image of the map Φ restricted to Γ (see Remark 3.16(b)) is shown to consist of μ-symmetric matrices in $\mathbb{P}(\sum_{k=1}^n \Bbbk M_k)$ of μ-rank at most two. Since a noncommutative quadratic form factors in at most two ways [VVe2, Theorem 2.8], the following result follows.

Theorem 3.17. [VVe2] *If the quadric system $\{q_1, \ldots, q_n\}$ associated to the GSCA, A, is normalizing and base-point free, then the number of isomorphism classes of left (respectively, right) point modules over A is equal to $2f_2 + f_1 \in \mathbb{N} \cup \{0, \infty\}$, where f_j denotes the number of matrices M in $\mathbb{P}\big(\sum_{k=1}^n \Bbbk M_k\big)$ such that μ-rank$(M) \leq 2$ and such that $\tau(M)$ factors in j distinct ways (up to a nonzero scalar multiple).*

Example. We consider a GSCA in [CaV, §5.3] with $n = 4$, where

$$q_1 = z_1z_2, \qquad q_2 = z_3^2, \qquad q_3 = z_1^2 - z_2z_4, \qquad q_4 = z_2^2 + z_4^2 - z_2z_3,$$
$$\mu_{23} = 1 = -\mu_{34}, \qquad (\mu_{14})^2 = \mu_{24} = -1, \qquad \mu_{13} = -\mu_{14},$$

This quadric system is normalizing and base-point free. By Theorem 3.15, the corresponding GSCA, A, is quadratic and regular of global dimension four, and is the $\Bbbk$-algebra on generators $x_1, \ldots, x_4$ with defining relations:

$$x_1x_3 = \mu_{14}x_3x_1, \qquad x_3x_4 = x_4x_3, \quad x_2x_3 + x_3x_2 = -x_4^2,$$
$$x_1x_4 = -\mu_{14}x_4x_1, \qquad x_4^2 = x_2^2, \qquad x_2x_4 - x_4x_2 = -x_1^2,$$

and has exactly five nonisomorphic point modules, two of which correspond to $q_1 = z_1z_2 = z_2z_1$. The other three point modules correspond to two quadratic forms in $\mathbb{P}\big(\sum_{k=1}^4 \Bbbk q_k\big)$ that have μ-rank one, namely

$$q_2 = z_3^2 \qquad \text{and} \qquad q_2 + 4q_4 = (z_2 - \frac{z_3}{2} + z_4)^2 = (-z_2 + \frac{z_3}{2} + z_4)^2,$$

where the latter quadratic form clearly factors in two distinct ways. Hence, A has a finite number of point modules even though two distinct elements of $\mathbb{P}\big(\sum_{k=1}^4 \Bbbk q_k\big)$ have μ-rank one.

4. Current and Future work

One of the goals for the work on GCAs and the introduction of GSCAs was to provide examples for quadratic algebras of global dimension three and four and this was the motivation for Section 3. Another goal is to provide candidates for generic quadratic regular algebras of global dimension four so as to contribute towards the classification of quadratic regular algebras of global dimension four. As it were, one of the families of GSCAs discussed in [CaV, §5] is a candidate for generic quadratic regular algebras of global dimension four. As discussed in [V], these algebras should have a point scheme with exactly twenty points and a one-dimensional line scheme.

Using techniques that involve Plücker coordinates, [ChV] analyzes the line scheme of this family of algebras. They find that the line scheme consists of one spatial elliptic curve, four planar elliptic curves, and a subscheme in a $\mathbb{P}^3$ consisting of the union of two nonsingular conics. In [TV], the authors analyze yet another family of algebras with a point scheme consisting of exactly twenty points and a one-dimensional line scheme and their analysis leads to a line scheme consisting of one spatial elliptic curve, one nonplanar rational curve, two planar elliptic curves, and two subschemes that is the union of a nonsingular conic and a line. This lead the authors to conjecture that the line scheme of the most generic quadratic regular algebra of global dimension four should be isomorphic to the union of two spatial elliptic curves and four planar elliptic curves.

Remark 4.1. 'Classically' defined Clifford algebras (see [L]) differ from the graded Clifford and graded skew Clifford algebras discussed here. One of the main differences is that the former are $\mathbb{Z}_2$-graded while the latter are $\mathbb{N}$-graded. Work done by [CK] has sought to bridge these differences.

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Padmini Veerapen
Department of Mathematics, P.O. Box 19408
Tennessee Tech University, Cookeville, TN 76019-0408

e-mail: pveerapen@tntech.edu