DEPARTMENT OF MATHEMATICS

TECHNICAL REPORT

40+ YEARS OF TEACHING AND THINKING ABOUT UNIVERSITY MATHEMATICS STUDENTS, PROOFS, AND PROVING: AN ABBREVIATED ACADEMIC MEMOIR

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40+ YEARS OF TEACHING AND THINKING ABOUT UNIVERSITY MATHEMATICS STUDENTS, PROOFS, AND PROVING: AN ABBREVIATED ACADEMIC MEMOIR

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INTRODUCTION: HOW WE GOT INTO THE FIELD

I plan to take you on a journey through how my husband John and I, who have PhDs in mathematics and spent our early academic years in pure mathematics, got into research in mathematics education. Along the way, I will discuss the kinds of research we have done, some challenges we have faced, and where we are in our thinking today. We have been mainly concerned with university students' learning of mathematical ideas and concepts, especially with proof and proving.

Before we got interested in mathematics education as a research subject, we were mathematicians teaching at least some upper-division and graduate mathematics courses using the Moore Method (Mahavier, 1999). Doing so gave us a lot of exposure to students' proving difficulties in courses like abstract algebra and topology. However, in other courses, like calculus, we lectured. The problem, as we saw it, was that despite our seemingly well-constructed and thoughtful lectures, students often had misconceptions and didn't perform as we wished. We wondered why. This was the germ of our interest in mathematics education research at the university level.

I finished my PhD in topological semigroups in l974, when academic positions in U.S. mathematics departments were scarce and the number of new PhDs in mathematics was perhaps at an all-time high. In order to obtain two academic teaching positions, we accepted posts at universities in Turkey (1974-1978) and later in Nigeria (1978-1985). While teaching mathematics at the University of the Bosphorus in Turkey, we wrote a paper analyzing university students' errors in logical reasoning (Selden & Selden, 1978). In that paper, we analyzed Turkish undergraduate university students' proofs submitted in a Moore Method course in abstract algebra. When we returned to the U.S. in 1985, we took up two academic teaching positions in the Department of Mathematics at Tennessee Technological University. When we heard of the 1987 *Cornell Conference on Misconceptions and Educational Strategies in Science and Mathematics*, we decided to "recast" our earlier paper on Turkish undergraduate university mathematics students' errors in logical reasoning in terms of misconceptions (Selden & Selden, 1987). This conference itself was an "eye opener." There we met Shlomo Vinner, who encouraged us to continue our research in mathematics education and to attend the conferences of the International Group for the Psychology of Mathematics Education (PME).

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[1] This paper is based on a plenary talk given at the 41st Annual Conference of the Canadian Mathematics Education Study Group in Montreal Canada, June 2017. Annie Selden is Professor Emerita of Mathematics from Tennessee Technological University and Adjunct Professor at New Mexico State University.

A little while later, in 1988, we attended the *Calculus for a New Century Symposium*, held at the National Academy of Sciences in Washington, DC. That was a "heady" time with all sorts of people, mathematicians, administrators, engineers, and so forth, thinking about calculus reform and how to teach more actively and better. Shortly thereafter, we conducted a small study (Selden, Mason, & Selden, 1989) on whether students, who had completed first-semester differential calculus with a grade of C, could solve nonroutine calculus problems, that is, problems they had not specifically been taught how to solve (Selden, Mason, & Selden, 1989). This led to two further studies of students' ability to solve nonroutine calculus problems (Selden & Selden,1994; Selden, Selden, Hauk, & Mason, 2000); one on first-semester A and B calculus students' ability to solve nonroutine calculus problems and another on first-semester A, B, and C calculus students', currently taking differential equations, ability to solve nonroutine calculus problems. This was our excursion into examining calculus students' ability to solve nonroutine problems. After that, probably because we had seen many students' difficulties with proving in Moore Method mathematics courses and in transition-to-proof courses, we switched our research to students' difficulties with proof and proving. The bulk of this paper is devoted to this work including our "unpacking" paper (Selden & Selden, 1995), our "validation" paper (Selden & Selden, 2003), and our "affect" (Selden, McKee, & Selden, 2010), all leading up to a discussion of our more recent theoretical work (Selden & Selden, 2015).

OUR THREE CALCULUS STUDIES

After having attended 1987 *Cornell Misconceptions and Educational Strategies in Science and Mathematics* and the 1988 *Calculus for a New Century Symposium*, we decided to conduct a study on 17 C first calculus students' ability to solve nonroutine calculus problems. This was partly inspired by the often expressed view of science and engineering professors that students just "can't do applications," where by applications they presumably meant the kinds of mathematical problems that come up in their courses. Our conjecture at the time was that applications, per se, were not the primary difficulty, rather we speculated that C calculus students weren't able to solve any nonroutine problems, whether in pure mathematics or applied subjects.

This led to our developing, in conjunction with our department chair, Alice Mason, a one-hour test of five nonroutine first calculus problems which we administered to volunteer C first calculus students (Selden, Mason, & Selden, 1989). The way we decided these were nonroutine for our calculus students was to ask the teachers of all the first calculus sections for that year whether they had taught their students how to do any of the problems we developed. None had. For example, the second of the five nonroutine problems was:

Does $x^{21} + x^{19} - x^{-1} + 2 = 0$ have any roots between -1 and 0? Why or why not?

We analyzed all responses to the problems, with the major finding being, "Notably, not a single student solved an entire problem correctly and most solution attempts relied heavily on earlier, more elementary, mathematics." (Selden, Mason, & Selden, 1989, p. 45). This led us to ask: How would A and B first calculus students do on the same problems? And, did they have the prerequisite knowledge?

This led to our second calculus study (Selden, Selden, & Mason, 1994), in which we administered the same nonroutine test and a subsequent routine test of prerequisite knowledge questions to 20 volunteer A and 19 volunteer B students who had completed first-semester differential calculus with those grades and were currently enrolled in second-semester integral calculus. For example, the matching prerequisite knowledge questions for the second nonroutine problem, given above, were:

> *If $f(x) = x^5 + x$, where is f increasing?*
>
> *If $f(x) = x^{-1}$, find $f'(x)$.*
>
> *If 5 is a root of $f(x) = 0$, at what point (if any) does the graph of $y = f(x)$ cross the x-axis?*

The major finding in this study was:

> Although they [the participating volunteer A and B students currently enrolled in second-semester integral calculus] performed slightly better on our test of nonroutine problems, two-thirds of the students failed to solve a single problem completely and more than 40% did not make substantial progress on a single problem. The routine test confirmed that these students possessed an adequate knowledge base of relevant calculus skills. (Selden, Selden & Mason, 1994, p. 19).

Convinced that students eventually do learn first-semester differential calculus, we decided to explore the "folk theorem" that students really learn a course in the next course that uses it. This time the same nonroutine and routine tests were administered to 28 students who had gotten A, B, or C in their differential calculus courses, but who were currently taking differential equations. The major finding was:

> More than half of these students [the participating volunteer students currently enrolled in differential equations] were unable to solve even one problem and more than a third made no substantial progress toward any solution. A routine test of associated algebra and calculus skills indicated that many of the students were familiar with the key calculus concepts for solving the nonroutine problems; nonetheless, students often used sophisticated algebraic methods rather than calculus in approaching the nonroutine problems. (Selden, Selden, Hauk, & Mason, 2000, p. 128).

Thus, one can have appropriate knowledge, but not think of using it. Why? We conjectured at the time that these students were accustomed to solving problems using worked examples from sections of their mathematics textbooks, and hence, had never had to think of how to start an arbitrary problem. Or, as Lithner (2008) later wrote, they were good at imitative reasoning, rather than creative reasoning.

While we conducted the third of our calculus studies not long after our second calculus study, as they say "life happens." We got busy with teaching and other research and put aside the data for about six years until we had an opportunity to mentor a young fellow mathematician, Shandy Hauk, into mathematics education research. This delay in analyzing our data enabled us to follow-up on what happened to at least some of the participating students, and to ask the question, "Does it matter whether students are able to solve nonroutine problems?" Perhaps surprisingly, our answer was both yes and no.

> No, because the students in this study were among the most successful at the university by a variety of traditional indicators, both at the time of the study and subsequently, yet half of them could not solve a single nonroutine problem. They had overall GPAs [grade point averages] of just above 3.0 [a B average] at the time of the study and almost double the graduation rate of the university as a whole. At least seven of them subsequently earned a master's degree and one a Ph.D. in mathematics. (p. 147).

And

> . . . yes, it does matter. Most mathematicians seem to regard this kind of [nonroutine] problem solving as a test of deep understanding and the ability to use knowledge flexibly. In addition, most applied problems that students will encounter later will probably be at least somewhat different from the exercises found in calculus (and other mathematics) textbooks. It seems likely that much original or creative work in mathematics would require novel problem solving at least at the modest level of the problems in this study. (p. 147).

We have continued to think about the difficulty that students have in "bringing to mind" factual knowledge that they possess when they might make good use of it during problem solving or theorem proving. We have tried planting "seeds" in the form of asking students to prove a useful, related technical set theory theorem early in our current "proofs course" in order to see if they would recall that technique later when it would be useful for proving a point-set topology theorem. So far this has not proved to be the case, and we have found no useful psychology research on how one might cause spreading activation of a useful sort to occur when needed. *Spreading activation* is a way that cognitive psychologists explain the phenomenon that a person is able to more quickly recall information about a topic once a related concept has been introduced. As with much of the psychology research literature, phenomena are studied in order to understand how they work "naturally," rather than how they might be harnessed to work in the service of some objective, such as solving mathematical problems or proving theorems.

A MOVE TO RESEARCH ON PROOF AND PROVING WITH OUR "UNPACKING" PAPER

After having worked on the three calculus studies, described above, and perhaps because we had taught Moore Method courses and transition-to-proof courses in a variety of ways, we had seen numerous student difficulties with proving. As a result, we switched our research interests to various aspects of proof and proving. Another reason for this may have been that, starting in 1989, we began attending the Advanced Mathematical Thinking Working Group sessions at the annual PME Conferences. That is how we became acquainted with the Group's work on the book, *Advanced Mathematical Thinking* (1991), which was already well underway when we joined the Group. However, somewhat later, Tommy Dreyfus agreed to edit a special follow-up issue on Advanced Mathematical Thinking for *Educational Studies in Mathematics*, and we were invited to submit a paper. For that special issue, we analyzed "found data" from tests and examinations given in several of our own transition-to-proof courses (Selden & Selden, 1995).

Before writing a bit about what is in that paper, it might be interesting for those who might think that publishing research in mathematics education is easy, that for our first attempt at writing this paper, we received rather lengthy reviews with the strong suggestion that we "totally re-conceptualize" it. For us, as mathematicians, who had published a number of papers in mathematics research journals and who had directed or co-directed PhDs in mathematics, this was a rather puzzling injunction. However, we went to work for perhaps half a year, attempting to do what the reviewers had advocated. We later heard, via the grapevine, that our paper was perhaps read more in advance of publication than later. However, according to recent Google citations, what we refer to as our "unpacking paper" has had 269 citations to date.

Our "unpacking" paper has both theoretical and empirical parts. The empirical part came from the analysis of our students' test and examination papers. The students, who had previous experience with logical translation in their transition-to-proof courses, were asked to translate (i.e., unpack) calculus statements, not all true, written in mathematical English into equivalent logical versions, using the symbols $\forall$, $\exists$, $\vee$, $\wedge$, $\neg$, $\rightarrow$, $\leftrightarrow$, and inserting all variables and quantifiers. One such mathematical statement was:

$$\textit{If } f \textit{ is defined at } a\textit{, then } \lim_{x\to a} f(x) \textit{ exists implies } f \textit{ is continuous at } a.$$

As it happens, this particular calculus statement, is not true and you might want to figure out why. Correct responses were not unique. However, one sample correct unpacking, given in the paper, was:

$$(\forall f \epsilon F)\ (\forall a \epsilon R)\ \{(f \textit{ is defined at } a) \rightarrow [(\lim_{x\to a} f(x) \textit{ exists}) \rightarrow (f \textit{ is continuous at } a)]\}.$$

Empirical results included the following: For simplified informal calculus statements, just 8.5% of all unpacking attempts were successful. For actual statements taken directly from calculus texts, this dropped to 5%. We inferred that these students would be *unable* to reliably relate informally stated theorems with the top-level logical structure of their proofs. Hence, these students could not be expected to construct proofs or evaluate their validity (Selden & Selden, 1995).

In addition to the above empirical findings, we introduced, or extended, some theoretical constructs, only some of which have since been taken up by the mathematics education research community. We extended the notion of *concept image* to *statement image*, because we thought that individuals could have images of both definitions of mathematical concepts and of theorems relating mathematical concepts. We introduced preliminary versions of the notions of *proof framework* and of *proof validation*, which have recently been taken up by at least some researchers in undergraduate mathematics education. By a *proof framework* we meant the portion of a proof that can be written from just the logical structure of the statement of the theorem. We have since expanded upon this idea and I will discuss that later. By *proof validation* we meant reading and checking a proof for correctness. In the appendix of our "unpacking paper," we included a hypothetical validation of the theorem that the sum of continuous functions is continuous. Tommy Dreyfus as editor said, at the time, that we could make the appendix into a separate paper, not for the special issue. But having had so much trouble getting our "unpacking paper" accepted, we declined.

In addition to the above theoretical distinctions, we also contrasted *formal* and *informal* mathematical statements. For example, *Differentiable functions are continuous*, is informal because a universal quantifier and a variable are omitted, and because it departs from the usual "if-then" form of the conditional. A corresponding formal version would be: *For all real-valued functions f, if f is differentiable, then f is continuous.* We see the former version as more memorable, while the later version facilitates students in beginning a proof by writing a proof framework.

It took us awhile to get our next study conducted and published. This may have been because, in addition to teaching, we did a lot of expository writing, which unfortunately, while needed, and a service to the community, is not well rewarded in academia. In particular, at the time of the calculus reform movement, circa 1987, some individuals in the mathematics community were awarded an NSF-grant to publish a newsletter, *UME Trends*: *News and Reports on Undergraduate Mathematics Education*, about reform efforts.[2] Ed Dubinsky was selected as editor and he asked us to write a column which he described as like "movie reviews" for mathematics education research papers. Thus, we developed the *Research Sampler* column, for which, we wrote 26 columns and 36 news/feature articles, published in *UME Trends* and *MAA Online* from 1989 through 2001. Since newsletters are considered "ephemeral," I do not think many of them are archived anywhere, but we have made our columns and news/feature articles available on www.academia.edu and www.researchgate.net, where some people have found them. Our next published research paper was concerned with the validation of proofs by undergraduates.

OUR VALIDATION STUDY

This was an exploratory study, published in the *Journal for Research in Mathematics Education*, of how eight mathematics and secondary education mathematics majors at the beginning of a transition-to-proof course validated (read and checked for correctness) four student-generated arguments purported to be proofs of a single theorem (Selden & Selden, 2003). The theorem was:

Theorem. For any positive integer n, if n^2 *is a multiple of* 3, *then n is a multiple of* 3.

We began the paper by expanding on what we meant by validation:

> A validation is often much longer and more complex than the written proof and may be difficult to observe because not all of it is conscious. Moreover, even its conscious part may be conducted silently using inner speech and vision. Validation can include asking and answering questions, assenting to claims, constructing subproofs, remembering or finding and interpreting other

[2]In 2000, Ed Dubinsky wrote a *FOCUS* article about the founding of *UME Trends*, which is available at http://www.math.kent.edu/~edd/FocusArticle.pdf

> theorems and definitions, complying with instructions (e.g., to consider or name something), and conscious (but probably nonverbal) feelings of rightness or wrongness. Proof validation can also include the production of a new text -- a validator-constructed modification of the written argument -- that might include additional calculations, expansions of definitions, or constructions of subproofs. Towards the end of a validation, in an effort to capture the essence of the argument in a single train-of-thought, contractions of the argument might be undertaken. (Selden & Selden, 2003, p. 5).

There was a theoretical part to this paper[3], but I will concentrate here on the empirical findings. The participating students were told that the four "proofs" they were to read and evaluate had been generated by students like themselves in a previous year. The first student-generated "proof" was the following:

> Proof: Assume that n^2 is an odd positive integer that is divisible by *3*. *That is* $n^2 = (3n + 1)^2 = 9n^2 + 6n + 1 = 3n(n + 2) + 1$. Therefore, n^2 is divisible by 3. Assume that n^2 is even and a multiple of 3. *That is* $n^2 = (3n)^2 = 9n^2 = 3n(3n)$. Therefore, n^2 is a multiple of 3. If we factor $n^2 = 9n^2$, we get $3n(3n)$; which means that n is a multiple of 3. ■ (Selden & Selden, 2003, p. 12).

We found at least 10 errors in the above "proof." Indeed, we presented it to our participants as the first argument to consider precisely because there are so many errors, with the notational ones being very easy to spot. However, all participants took a good deal of time reading and trying to make sense of the above argument, with one participant spending 15 minutes reading and rereading, before finally deciding it was a proof, except for one minor notational error.

The empirical findings on validation included the following. Participants' correct judgments on whether a given argument was, or was not, a proof went from 46% correct (i.e., essentially chance level) to 81% correct, after having considered and reconsidered the arguments. "Most of the errors detected were of a local/detailed nature rather than a global structural nature." (p. 24). To our knowledge, this was the first published paper to investigate students' proof validations, and later studies have confirmed this result. For example, the eye-tracking study by Inglis & Alcock (2012) found that "compared with mathematicians, undergraduate students spend proportionately more time focusing on 'surface features' of arguments." (p. 358).

At the end of the interview, participants were also asked debrief questions such as, (1) When you read a proof is there anything different you do, say, than in reading a newspaper? and (2) Specifically, what do you do when you read a proof? From their responses, we concluded that:

> What students say about how they read proofs seems to be a poor indicator of whether they can actually validate proofs with reasonable reliability. They tend

[3] For example, we took the view that the meaning resides in the proof texts themselves. Much like, for Martin Luther, the meaning of Scripture could be found in a deeper reading of the Bible (Selden & Selden, 2003, p.6, Footnote 2).

> to "talk a good line." They say that they "check proofs step-by-step, follow arguments logically, generate examples, and make sure the ideas in a proof make sense." (Selden & Selden, 2003, p. 27)

The "unpacking" and "validation" papers, described above, while introducing some theoretical concepts such as proof frameworks and proof validation, were largely empirical. After that, we became more theoretical in our research and writings.

MOVING TOWARD A THEORETICAL PERSPECTIVE ON PROOF AND PROVING

The next research paper, which we sometimes refer to as our "affect" paper, appeared in a special issue of the *International Journal of Mathematical Education in Science and Technology* (*iJMEST*). It considered the role of consciousness in the proving process and introduced the ideas of <*situation, action*> pairs, *behavioral schemas*, and *non-emotional cognitive feelings* (Selden, McKee, & Selden, 2010). We wrote:

> We see (at least the conscious part of) cognition in general, and the proving process in particular, as a sequence of mental and physical actions, such as writing or thinking a line in a proof, drawing or visualizing a diagram, reflecting on the results of earlier actions, or trying to remember an example. Many such actions appear to be guided by small 'habits of mind' that often link a particular recognized situation to a particular action. Such <situation, action> pairs, or habits of mind, can reduce the burden on working memory.
>
> As a person gains experience, much of proof construction appears to be separable into sequences of small parts, consisting of recognizing a situation and taking a mental or physical action. Actions which once may have required a conscious warrant become automatically linked to triggering situations. From a third-person, or outside, perspective these regularly linked <situation, action> pairs might be regarded as small 'habits of mind' [14]. On the other hand, taking a first-person, inside, or psychological perspective, they are lasting mental structures that we have called behavioral schemas. (Selden, McKee, & Selden, 2010, p. 204).

An example of a *non-emotional cognitive feeling* would be a feeling of being on the right track. A *behavioral schema* that might be invoked during proving begins with a situation. For example, one might be starting to prove a statement having a conclusion of the form p or q. This would be the situation. Having encountered this situation many times before, one might readily write into the proof "Assume *not* p" and proceed to attempt to prove q or vice versa. While this action can be warranted by logic (if *not* p then q, is equivalent to, p or q), there would no longer be a need to do so.

In that "affect" paper, we presented a six-point theoretical sketch of the genesis and enactment of behavioral schemas, which I paraphrase below:

1) Within very broad contextual considerations, behavioral schemas are immediately available. They do not normally have to be recalled, that is, searched for and brought to mind.

2) Simple behavioral schemas operate outside of consciousness. One is not aware of doing anything immediately prior to the resulting action – one just does it.

3) Behavioral schemas produce immediate action, which may lead to subsequent action. One becomes conscious of the resulting action as it occurs or immediately afterwards.

4) One cannot "chain together" behavioral schemas in a way that functions entirely outside of consciousness and produces consciousness of only the final action.

5) An action due to a behavioral schema depends on conscious input, at least in large part.

6) Behavioral schemas are acquired through practice. To acquire a beneficial schema a person should actually carry out the appropriate action correctly a number of times – not just understand its appropriateness. Changing a detrimental behavioral schema requires similar, perhaps longer, practice. (Selden, McKee, & Selden, 2010, pp. 205-206).

Behavioral schemas can be detrimental, as well as beneficial. An example of a detrimental behavioral schema that students often learn tacitly is that $\sqrt{a^2 + b^2}$ equals $a + b$. Showing a counter example is ineffective -- almost any two numbers substituted for a and b show this is false. However, students persist. We would not refer to this as a misconception, but rather characterize it as a "misaction." Perhaps the most interesting detrimental schema, reported in our "affect" paper, was the case of a mathematics graduate student, Sofia, in our "proofs course" who had developed an "unreflective guess" schema, which we also thought of as "grasping at straws," Enacting this schema often prevented Sofia from making progress on constructing proofs. Since one such unreflective guess often led to another for Sofia, we wanted to rid her of this "bad habit," so we suggested substitute actions such as**:** draw a figure, look for inferences from the hypotheses, reflect on everything done so far, or even do something else for a while. (p. 212). By the end of the semester, we felt Sofia had made much progress. To substantiate this, we included in the paper the following:

> As the course ended, our intervention of directing Sofia to do something else, whether it be draw a diagram or review her notes, was beginning to show promise. For example, on the in-class final examination Sofia proved that if f, g, and h are functions from a set to itself, f is one-to-one, and $f \circ g = f \circ h$, then $g = h$. Also on the take-home final, except for a small omission, she proved that the set of points on which two continuous functions between Hausdorff spaces agree is closed. This shows Sofia was able to complete the problem-centred parts of at least a few proofs by the end of the course, and suggests her 'unreflective guess' behavioural schema was weakened. (Selden, McKee,& Selden, 2010, p. 212).

AN EXCURSION INTO RESEARCH ON READING

We next investigated how precalculus and calculus students read their textbooks (Shepherd, Selden, & Selden, 2012). This study was conducted with a mathematician colleague, Mary Shepherd, who wished to be mentored into the field. The students in this study had high ACT[4] mathematics and reading scores and did much of what good readers do. The selected participants were invited to read passages, a little ahead of where they currently were reading for the course, but which their teacher judged were accessible to them. The research question we asked was: Could they work straightforward tasks associated with the reading soon after reading passages explaining, or illustrating, how the tasks should be carried out, and with those passages still available to them?

Our empirical findings included: Only three of the eleven volunteer students could independently work at least half of the tasks. Why? Their difficulties seemed to arise from: (1) insufficient sensitivity to, or inappropriate response to, confusion or error; (2) inadequate or incorrect prior knowledge; and (3) insufficient attention to details, often due to mind wandering. (Shepherd, Selden, & Selden, 2012, p. 238).

We wondered: Why are mathematicians good readers of mathematical texts? While we did not go on to investigate this, our colleague, Mary Shepherd did. She explored how mathematicians read mathematical material unfamiliar to them (Shepherd & van de Sande, 2014). Being a mathematician herself, she selected the beginning section of a volume on differential geometry, a subject her participants were unfamiliar with, but was in her area of expertise. Perhaps the most interesting finding of this study, was that the mathematicians, but not the graduate student participants, often engaged in what she called *reading-the-meaning.* For example, when coming to the definition of a metric space, the mathematicians quickly noted this without reading the words or symbols – it was as if they were seeing a familiar icon. In addition, the authors proposed a framework for reading mathematical exposition from novice to intermediate to expert (Shepherd & van de Sande, 2014, p. 85).

Of interest, to young researchers perhaps, is that it took 10 years from my first answering Mary Shepherd's email listserv request for help until the publication of our joint paper. Not knowing the reading comprehension research literature, Mary and I delved into it for about two years and corresponded via email about it. Then we, with John, met to design the study and Mary collected the data. Then, for awhile, I got sick and the study was set aside. After we finally submitted our manuscript, the journal took about a year to get back to us and rejected it. Then the three of us took what we thought were the legitimate criticisms of the reviews, rewrote the paper, and submitted it to *Mathematical Thinking and Learning*. This time we got back a "revise and resubmit," rewrote the paper again, and eventually it was published.

BACK TO PROOF AND PROVING: OUR CURRENT PERSPECTIVE ON PROOF CONSTRUCTION

Much of our current theoretical perspective was detailed in our *RUME Proceedings* paper (Selden & Selden, 2015), which was given an honorable mention by the Mathematical Association of America's Special Interest Group on Research in Undergraduate Mathematics Education. It was

[4] Most students in the U.S. are required to have at least minimum scores (set by each university) on a national reading comprehension test and a national mathematics test, either the ACT or the SAT, as well as other qualifying materials in order to be admitted to the university. The ACT tests are provided by American College Testing, Inc.

an attempt to "weave together" all of our previous work on proof, together with more recent research from 10+ years of teaching our inquiry-based "proofs course". That course is taught mainly to beginning mathematics graduate students who feel they need help with proof construction. It is taught entirely from notes with students constructing original (to them) proofs and receiving, sometimes extensive, critiques in class. Topics very briefly covered include sets, functions, real analysis, abstract algebra (in the form of semigroups), and if time permits, some point-set topology. A major aim of the course is to facilitate students' learning through experiences constructing as many different kinds of proofs as possible. Another important aim of the course is to have students learn to write proofs acceptable to their other professors.

We video and analyze all classes and many planning sessions. As a consequence, we have developed a theoretical perspective of the proving process that includes: (a) mathematical aspects and (b) psychological aspects. First, I will discuss some of the mathematical aspects of our theoretical perspective including the genre of proofs and the structure of proof texts.

MATHEMATICAL ASPECTS: THE GENRE OF PROOFS

Our thinking and research on this topic actually began around Summer 1999 when we attended the Institute for Advanced Study/Park City Mathematics Institute (PCMI), which is designed for mathematics researchers, post-secondary students, and mathematics educators at the secondary and post-secondary levels. We had previously observed in our teaching that students sometimes find the manner in which proofs are written perplexing. That is, it is often at variance with other genres of writing, and we had identified some significant features that generally occur in proofs.

While at PCMI, we interviewed volunteer mathematicians about what they thought about some of our conjectured features of proofs, while they were looking at one of their own published mathematics papers. These features are indicated below.

1. Proofs are not reports of the proving process.
2. Proofs contain little redundancy.
3. Symbols are (generally) introduced in one-to-one correspondence with mathematical objects.
4. Proofs contain only minimal explanations of inferences, that is, warrants are often left implicit.
5. Proofs contain only very short overviews or advance organizers.
6. Entire definitions, available outside the proof, are not quoted in proofs.
7. Proofs are "logically concrete" in the sense that quantifiers, especially universal quantifiers, are avoided where possible. (Selden & Selden, 2013b).

None of the above features is very surprising for mathematicians, especially the first one. However, we know of no other study on this topic. But as has so often happened with us, life and other academic duties interfered, and the data were put aside and not actually published until the book originating from the Symposium in Honor of Ted Eisenberg's retirement was published. (Selden & Selden, 2013b).

STRUCTURES OF PROOFS

Some of our more recent thinking on this was incipient in our earlier papers (e.g., Selden & Selden, 1995). A proof text can be divided into a *formal-rhetorical* part and a *problem-centered part*. The *formal-rhetorical* part is the part that depends only on unpacking the logical structure of the statement of the theorem, associated definitions, and earlier results. In general, this part does not depend on a deep understanding of the concepts or genuine problem solving in the sense of Schoenfeld (1985). We call the remaining part of a proof the *problem-centered* part. It does depend on problem solving, intuition, heuristics, and understanding the concepts involved. (Selden & Selden, 2011).

A feature that can help write the formal-rhetorical part of a proof is what we have called a *proof framework*, an idea we introduced earlier in our "unpacking" paper. However, we have since expanded on this idea to include several different kinds, and in most cases, both a *first*- and a *second-level* framework. We have detailed this in a recently published *PRIMUS* paper (Selden, Selden, & Benkhalti, 2017). Briefly, given a theorem of the form "For all real numbers x, if $P(x)$ then $Q(x)$," a *first-level proof framework* would be "Let x be a real number. Suppose $P(x)$. … Therefore, $Q(x)$," with the remainder of the proof ultimately replacing the ellipsis. A *second-level framework* can often be obtained by "unpacking" the meaning of $Q(x)$ and putting the second-level framework between the lines already written for the first-level framework. Thus, the proof would "grow" from both ends toward the middle, instead of being written from the top down.

To write a second-level framework, one often needs to convert formal mathematical definitions and previously proved results into their *operable interpretations* – something that we initially found surprising. For example: Given a function $f: X \rightarrow Y$ and $A \subseteq Y$, one defines $f^{-1}(A) = \{x \in X \mid f(x) \in A\}$. An operable interpretation would say, "If you have $b \in f^{-1}(A)$, then you can write $f(b) \in A$ and vice versa." One might think, as we previously often did, that translation into an operable form would be unnecessary or easy especially because the symbols in $\{x \in X \mid f(x) \in A\}$ can be translated into words in a one-to-one way, but for some students it requires help and practice.

THE NEED FOR PREVIOUS RESULTS—PROOFS OF TYPES 0, 1, 2, 3

In order to enhance the possibility of student successes in our inquiry-based "proofs course", we have classified theorems of increasing difficulty in our course notes which consist of statements of theorems, definitions, and questions. While we had been thinking about proof difficulty for some time, I believe we first discussed some of these proof types in an invited article for a special issue of *The Mathematics Enthusiast* (Selden & Selden, 2013a). Our current classification into proof types is as follows:

- *Type* 0 often follows immediately from definitions.
- *Type 1* may need a result in the notes.

- *Type 2* needs a lemma, not in the notes, but relatively easily to discern, formulate, and prove.
- *Type 3* should have at least one of discern, formulate, or prove be difficult.

Here is an example of a Type 3 proof of a theorem from our course notes: *A commutative semigroup S with no proper ideals is a group*, when one is provided only the definitions of semigroup and ideal. One first needs to observe that, for $a \in S$, aS is an ideal, so $aS=S$. This, in turn, implies that equations of the form $ax=b$ are solvable for any $a \in S$ and any $b \in S$. We could have formulated these two facts as lemmas in advance of the theorem statement, but we chose not to. If one formulates and proves these two lemmas, then, using some clever instantiations of the equation $ax=b$, one can obtain an identity and inverses, and conclude S is a group. At last count, in our "proofs course", only two of 74 students, after much hard work, have been able to prove this theorem on their own.

One of the reasons for classifying proofs into types according to difficulty is to be able to estimate, in advance, which students to call on to present their proof attempts at the board. We feel that, if a proof is too easy for a student or if a proof is too hard for a student, then probably nothing will be learned by that student or the class. Worst of all, being called upon to present a proof attempt that one does not think is worth discussing, may lower that student's sense of *self-efficacy*, a topic addressed below.

THE NEED FOR UNGUIDED EXPLORATION

In constructing some proofs, one may reach a point where there is no "natural" way forward. One has come to an impasse, that is, colloquially, one is "stuck." In what we call *unguided exploration*, one may need to find, or define, an object and prove something about it, with no idea of its usefulness, that is, one may need to "explore" the situation to get an insight.

For example, in proving the above Type 3 semigroup theorem, this kind of exploration, followed by a helpful insight, can happen at least twice. The first helpful insight comes when one notes that aS is an ideal, and hence, $aS=S$. The next helpful insight comes when one sees that the set equation, $aS=S$, implies that element equations of the form $ax=b$ are solvable for any $a \in S$ and any $b \in S$. We feel that several kinds of proofs of Type 3 may require considerable perseverance and self-efficacy. We try to engender this in students by arranging for early proving successes, followed by assigning proofs of increasing difficulty. Discussing associated heuristics well before a target theorem arises may also be useful.

THE NEED TO UNPACK THE LOGICAL STRUCTURE OF A THEOREM STATEMENT

We made the distinction between informal and formal mathematical statements in our "unpacking" paper, mentioned briefly above. *Informally stated theorems* are commonplace in everyday mathematics. They are not ambiguous or ill-formed because widely understood, but rarely articulated, conventions permit their precise interpretation by mathematicians and less reliably by students. Such informally stated theorems can be memorable and perhaps easily brought to mind, but may be difficult to unpack and prove, especially for beginners.

We have found that proof frameworks are relatively easy to write for theorems stated in the customary “if, then” format. Thus, early on in our notes, we write theorem statements in this format. However, we know that students must eventually be able to unpack informally stated theorems into their “if, then” format in order to decipher what the theorem is stating, what the hypotheses are, and what the conclusions are in order to begin the process of proof construction by writing a proof framework.

PSYCHOLOGICAL ASPECTS OF THE PROVING PROCESS

As mentioned above, we view proof construction as a sequence of actions which can be physical (e.g., writing a line of the proof or drawing a sketch) or mental (e.g., changing one’s focus from the hypothesis to the conclusion, trying to recall a theorem, or bringing up a feeling). The sequence of actions that eventually leads to a proof is usually considerably longer than the final proof and is often not constructed from the top down. Somewhat surprising to us, we once had a mathematics education graduate student from the School of Education, who did not know that proofs were not written from the top-down. Her recollection from a prior real analysis course was that the professor wrote proofs from the top-down in lectures, so she had just assumed that was the way proofs are constructed.

Some of what I write next may seem similar to what I wrote above. That is, no doubt, because theoretical perspectives develop slowly over time and our current perspective grew out of our earlier theoretical observations and empirical studies.

SITUATION-ACTION LINKS, AUTOMATICITY, AND BEHAVIORAL SCHEMAS

If, during several proof constructions in the past, similar situations have corresponded to similar reasoning leading to similar actions, then a *link* may be learned between them, so that another similar situation evokes the corresponding action in future proof constructions without the need for the earlier intermediate reasoning. Using such *situation-action links*, or <*situation, action*> pairs as we called them earlier, strengthens them, and after sufficient experience/practice, they can become overlearned and automated, and hence, become *behavioral schemas*. These are the same behavioral schemas that I described above and whose six properties I mentioned.

There are cognitive advantages to invoking automaticity appropriately during proof construction. So we aim to help our students convert System 2 (S2) cognition into System 1 (S1) cognition where appropriate. *S2 cognition* is slow, conscious, effortful, evolutionarily recent, and calls on considerable working memory. In contrast, *S1 cognition* is fast, unconscious, automatic, effortless, evolutionarily ancient, and places little burden on working memory. (Stanovich & West, 2000). Converting S2 cognition into S1 cognition conserves working memory, a precious resource.

In discussing automaticity, we are largely depending on the work of social psychologists (e.g., Bargh & Chartrand, 2000). In general, according to Bargh (1994), an individual executing an automated action tends to: (1) Be unaware of any needed mental process; (2) Be unaware of intentionally initiating the action; (3) Execute the action while putting little load on working memory; and (4) Find it difficult to stop or alter the action. However, not necessarily all four tendencies occur in every situation. We feel that the first three of these tendencies, appropriately harnessed during proof construction, would help conserve students' working memory for the truly hard parts of proofs.

We view behavioral schemas as belonging to a person's knowledge base. They can be considered as partly conceptual knowledge (recognizing and interpreting the situation) and partly procedural knowledge (doing the action), and as related to Mason and Spence's (1999) idea of "knowing-to-act in the moment." We aim to encourage and develop beneficial behavioral schemas for proving in our students and discourage, and hopefully extinguish, detrimental behavioral schemas for proving, such as Sofia's detrimental "unreflective guess" schema, described above.

As a result of enacting beneficial proving behavioral schemas advantageously, students might simply not have to think quite so deeply about certain portions of the proving process, and might have more working memory available for the harder parts of a proof. However, helping students develop beneficial behavioral schemas is no easy task, because the process of learning a behavioral schema can often be implicit, although the situation and the action are at least, in part, conscious. That is, an individual can acquire a behavioral schema without being aware that it is happening. Indeed, such unintentional, or implicit, learning happens frequently (e.g., Cleeremans, 1993).

NON-EMOTIONAL COGNITIVE FEELINGS IN PROOF CONSTRUCTION

We are particularly interested in the kinds of affect that might occur during proof construction, and have considered feelings, especially *non-emotional cognitive feelings*, as mentioned above. Often the terms "feelings" and "emotions" are used more or less interchangeably, perhaps because both appear to be conscious reports of unconscious mental states, and each can, but need not, engender the other. However, we follow Damasio (2003) in separating feelings from emotions because emotions are expressed by observable physical characteristics, such as temperature, facial expression, blood pressure, pulse rate, perspiration, and so forth, while feelings are not.

Here are some example of the kinds of non-emotional cognitive feelings we are interested in: (1) a feeling of knowing that one has seen a theorem useful for constructing a proof, but which one is not able to bring to mind at the moment; (2) a feeling of familiarity; and (3) a feeling of rightness. Such *non-emotional cognitive feelings* can guide cognitive actions. For example, these can influence whether one continues a search for a solution or a proof or aborts it.

Feelings seem to be summative in nature and pervade an individual's whole field of consciousness at any particular moment. For example, one can have a feeling of unease in the midst of concentrating on developing a proof or solving a problem. Finally, we conjecture that feelings may eventually be found to play a larger role in proof construction than indicated above, because they can provide a direct link between the conscious mind and the structures and possible actions of the nonconscious mind, which can process many streams of information in parallel.

THE ROLE OF SELF-EFFICACY IN PROOF CONSTRUCTION

In order to prove harder theorems--ones with a substantial problem-centered part--students need to persist in their efforts, and such persistence can be facilitated by a *sense of self-efficacy (*Selden & Selden, 2014). According to Bandura (1995), self-efficacy is "a person's belief in his or her ability to succeed in a particular situation." Of developing a sense of self-efficacy, Bandura (1994) stated that "The most effective way of developing a strong sense of self-efficacy is through mastery experiences," that performing a task successfully strengthens one's sense of self-efficacy. Also, according to Bandura, "Seeing people similar to oneself succeed by sustained effort raises observers' beliefs that they too possess the capabilities to master comparable activities to succeed." Bandura's ideas "ring true" with our past experiences as mathematicians teaching courses by the Modified Moore Method.

Some of the ways we attempt to develop students' self-efficacy are by: (1) letting them know early on that the raison d'etre of the course is to help them with their proof writing, (2) arranging for early proving successes, and (3) slowly increasing the difficulty of the proofs in the course notes.

THE DEVELOPMENT AND USE OF LOCAL MEMORY

In constructing a proof of some complexity, often much more relevant information is brought to mind than can be held in one's working memory. When such information is lost from consciousness, it may remain partially activated and easily accessed. We refer such partially activated information as *local memory.* While we have experienced such easily accessed information during our own mathematical research, as well as its loss when too many days or weeks have passed by, we know of little research about its development, maintenance, and uses even in the psychology literature. Nonetheless, it seems to us that conscious thought can sometimes influence the activation of such information, that is, help bring something to mind. We have observed of ourselves, when attempting an intricate complex proof, that a considerable amount of information is generated, but cannot all be kept in mind; however, it is easily recalled. We speculate that many mathematicians do this when conducting their own research. We feel local memory is an area worthy of future research.

What good is a theoretical perspective if one cannot use it for anything? Below, I discuss how we use our perspective to construct proofs and analyze students' proof attempts. I also indicate how we use our perspective in designing our "proofs course".

USING OUR PERSPECTIVE TO CONSTRUCT AND ANALYZE PROOFS

First, I show how one can construct a sample correct proof of a theorem using a proof framework and operable interpretations of definitions. Then I use our perspective to analyze an incorrect student proof attempt of the same theorem.

CONSTRUCTION OF A SAMPLE CORRECT PROOF

The theorem I consider comes from the semigroup portion of our "proofs course". We begin with the statement of the theorem and its proof, showing both the first- and second-level proof frameworks. The second-level proof framework, lines [3] and [4] below, comes from unpacking the meaning of "commutative."

Theorem. *Let S be a semigroup with an identity element e. If, for all s in S, ss = e, then S is commutative.*

Proof:

[1] Let S be a semigroup with identity e. Suppose for all $s \in S$, $ss = e$.
[3] Let a, b be elements in S.

. . .

[4] Thus $ba = ab$.
[2] Therefore, S is commutative. QED.

I now continue with the rest of the proof, filling in the ellipsis above. This may involve a bit of "messing around" with equations, that is, it may require quite a bit of exploration. A prover of this semigroup theorem can write many unhelpful equations before coming upon useful equations that result in a proof. After selecting parts of his/her exploration and rearranging them, the remaining part of the could have been written as lines [5], [6], and [7] below.

Theorem. *Let S be a semigroup with an identity element e. If, for all s in S, ss = e, then S is commutative.*

Proof:

[1] Let S be a semigroup with identity e. Suppose for all $s \in S$, $ss = e$.
[3] Let a, b be elements in S.
[5] Now, $abab = e$, so $(abab)b = eb = b$.
[6] But $(abab)b = aba(bb) = abae = aba$.
[7] So, $b = aba$, so $ba = (aba)a = ab(aa) = abe = ab$.
[4] Thus $ba = ab$.
[2] Therefore, S is commutative. QED.

We are not claiming that teaching students to write proof frameworks is a panacea. Rather, being able to write a complete proof framework exposes the "real problem" to be solved during the proof construction process. We are also not claiming that mathematicians write proofs this way—only that they will accept the results of proofs written this way.

A STUDENT'S INCORRECT PROOF ATTEMPT OF THE SAME THEOREM

First, I will present what the student wrote and submitted as a proof of the theorem. Then I will analyze it using our theoretical perspective, looking for beneficial actions *not taken* and detrimental actions *taken*. Recall that the theorem is:

Theorem. *Let S be a semigroup with an identity element e. If, for all s in S, ss = e, then S is commutative.*

The student's incorrect proof attempt, including scratch work, is given below:

Let S be a semigroup with an identity element, e. Let $s \in S$ such that $ss = e$. Because e is an identity element, $es = se = s$. Now, $s = se = s(ss)$.	Scratchwork 7.1: A semigroup is called commutative or Abelian if, for each a and $b \in S$, $ab = ba$.

Since S is a semigroup, $(ss)s = es = s$. Thus $es = se$. Therefore, S is commutative. QED.	7.5: An element e of a semigroup S is called an identity element of S if, for all $s \in S$, $es = se = s$.

ANALYSIS OF THE STUDENT'S INCORRECT PROOF ATTEMPT

First, we look at beneficial actions *not taken*. The second sentence should have been "Suppose for **all** $s \in S$, $ss = e$." With this change to include "all $s \in S$," the first-level framework would have been correct. In addition, the student did not produce a second-level framework by introducing arbitrary a and b at the top, followed by "Then $ab = ba$" right above the conclusion.

Had the student written the correct second sentence and taken the above actions, the situation would have been appropriate for exploring and manipulating an object such as *abab*. We think that such exploration calls for some self-efficacy, but can lead to a correct proof.

Next, we consider what the student wrote "in the middle," and analyze it line-by-line. It was:

> Because e is an identity element, $es = se = s$.
> Now, $s = se = s(ss)$.
> Since S is a semigroup, $(ss)s = es = s$.
> Thus $es = se$.

The first line of "the middle" above violates the mathematical norm of not including definitions, that can easily be found outside the proof, in the proof. Also, it does not move the proof forward. The next three lines are not wrong, but do not move the proof forward because to prove commutativity, one needs two arbitrary elements. We consider these actions detrimental because they can convince the student that he/she has accomplished something when that is not the case. This completes the analysis of the incorrect student proof attempt.

HOW WE USE OUR PERSPECTIVE IN DESIGNING OUR "proofs course"

I only mention a couple of things we do in designing and teaching the course. We want students to have early successes, so they gain a sense of self-efficacy. Therefore, we try to have relatively easy theorems at the beginning of our notes and gradually increase their difficulty. At the beginning of the course, we have students practice writing proof frameworks (without necessarily having to write complete proofs). There are several kinds of proof frameworks, but I have only demonstrated one kind. For example, other proof frameworks can involve proofs by cases or proofs by parts. There is yet another proof framework for a proof by contradiction.

As for operable interpretations of definitions which are needed to write second-level proof frameworks, we have used handouts (with definitions on the left side of the paper and operable interpretations on the right side). The idea of doing this was to have students make "flash cards" using these handouts and practice these operable interpretations. Unfortunately, while these handouts were somewhat helpful, we found some students, who despite having these handouts available on our examinations, still did not use them appropriately in writing their proofs.

TEACHING AND FUTURE RESEARCH CONSIDERATIONS

We believe this perspective on proving, using situation-action links and behavioral schemas, together with information from psychology, is mostly new to the field. Thus, it is likely to lead to additional insights and teaching interventions. This brings up the question of priorities. For example: Which proving actions of the kinds discussed above are most useful for mid-level university mathematics students to automate when they are learning to construct proofs?

Since every proof can be constructed using a proof framework, we consider constructing proof frameworks as a reasonable place to start. Doing so will get students started on proofs, as well as know where they are headed. This is preferable to staring at a blank piece of paper and not knowing what to do. Furthermore, we are not claiming that mathematicians write proofs this way -- only that professors will accept the results of writing proofs this way.

For students to have early successes and build self-efficacy, one can begin with more formally stated "if-then" theorems and later go to more informally stated theorems, which are harder to unpack. Furthermore, we have observed that some students do not write a second-level proof framework, perhaps because they have difficulty unpacking the meaning of the conclusion. Consequently, we are working on having students develop and use operable interpretations of definitions.

SUMMARY AND CONCLUSION

In this paper, I first briefly mentioned: (1) how we got into mathematics education research; (2) our early calculus studies; (3) our "unpacking" paper; (4) our "validation" paper; (5) our "affect" paper; and (6) our "reading" paper Then, I discussed at greater length our current theoretical perspective on proof construction, how we use it to analyze student proof attempts, and how we use it in designing our "proofs course", as well as mentioning a few research questions. We would be pleased if others considered some of these ideas.

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