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A CLASSIFICATION OF CLIFFORD ALGEBRAS AS IMAGES OF GROUP ALGEBRAS OF SALINGAROS VEE GROUPS

R. Abłamowicz, M. Varahagiri, A. M. Walley

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TENNESSEE TECHNOLOGICAL UNIVERSITY
Cookeville, TN 38505

A Classification of Clifford Algebras as Images of Group Algebras of Salingaros Vee Groups

Rafał Abłamowicz, Manisha Varahagiri and Anne Marie Walley

Abstract. The main objective of this work is to prove that every Clifford algebra $C\ell_{p,q}$ is $\mathbb{R}$-isomorphic to a quotient of a group algebra $\mathbb{R}[G_{p,q}]$ modulo an ideal $\mathcal{J} = (1 + \tau)$ where τ is a central element of order 2. Here, $G_{p,q}$ is a 2-group of order 2^{p+q+1} belonging to one of Salingaros isomorphism classes N_{2k-1}, N_{2k}, Ω_{2k-1}, Ω_{2k} or S_k. Thus, Clifford algebras $C\ell_{p,q}$ can be classified by Salingaros classes. Since the group algebras $\mathbb{R}[G_{p,q}]$ are $\mathbb{Z}_2$-graded and the ideal $\mathcal{J}$ is homogeneous, the quotient algebras $\mathbb{R}[G]/\mathcal{J}$ are $\mathbb{Z}_2$-graded. In some instances, the isomorphism $\mathbb{R}[G]/\mathcal{J} \cong C\ell_{p,q}$ is also $\mathbb{Z}_2$-graded. By Salingaros Theorem, the groups $G_{p,q}$ in the classes N_{2k-1} and N_{2k} are iterative central products of the dihedral group D_8 and the quaternion group Q_8, and so they are extra-special. The groups in the classes Ω_{2k-1} and Ω_{2k} are central products of N_{2k-1} and N_{2k} with $C_2 \times C_2$, respectively. The groups in the class S_k are central products of N_{2k} or N_{2k} with C_4. Two algorithms to factor any $G_{p,q}$ into an internal central product, depending on the class, are given. A complete table of central factorizations for groups of order up to $1,024$ is presented.

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1. Introduction

Salingaros [17–19] defined and studied five families N_{2k-1}, N_{2k}, Ω_{2k-1}, Ω_{2k} and S_k of finite 2-groups related to Clifford algebras $C\ell_{p,q}$. For each $k \geq 1$, the group N_{2k-1} is a central product $(D_8)^{\circ k}$ of k copies of the dihedral group D_8 while the group N_{2k} is a central product $(D_8)^{\circ(k-1)} \circ Q_8$ of $k - 1$

copies of D_8 with the quaternion group Q_8 [1]. The groups N_{2k-1} and N_{2k} are extra-special. Salingaros showed that $\Omega_{2k-1} \cong N_{2k-1} \circ D_4$, $\Omega_{2k} \cong N_{2k} \circ D_4$, and $S_k \cong N_{2k-1} \circ C_4 \cong N_{2k} \circ C_4$, where C_2 and C_4 are cyclic groups of order 2 and 4, respectively, and $D_4 = C_2 \times C_2$ (cf. [6]). Chernov [7] observed that a Clifford algebra $C\ell_{p,q}$ could be obtained as a homomorphic image of a group algebra $\mathbb{R}[G]$ assuming there exists a suitable finite 2-group with generators fulfilling certain relations. As an example, he showed that $C\ell_{0,2} \cong \mathbb{R}[Q_8]/\mathcal{J}$ while $C\ell_{1,1} \cong \mathbb{R}[D_8]/\mathcal{J}$, where in each case $\mathcal{J}$ is an ideal generated by $1+\tau$ for a central element τ of order 2 and the isomorphisms are $\mathbb{R}$-algebra isomorphisms. Walley [21] showed that $C\ell_{2,0} \cong \mathbb{R}[D_8]/\mathcal{J}$, which was to be expected since $C\ell_{1,1} \cong C\ell_{2,0}$ (as $\mathbb{R}$-algebras). Walley also showed that Clifford algebras of dimension eight in three isomorphism classes can be represented as $C\ell_{0,3} \cong \mathbb{R}[G_{0,3}]/\mathcal{J}$, $C\ell_{2,1} \cong \mathbb{R}[G_{2,1}]/\mathcal{J}$, $C\ell_{1,2} \cong \mathbb{R}[G_{1,2}]/\mathcal{J}$, and $C\ell_{3,0} \cong \mathbb{R}[G_{3,0}]/\mathcal{J}$ where $G_{0,3} \in \Omega_2$, $G_{2,1} \in \Omega_1$, and $G_{1,2} \cong G_{3,0} \in S_1$. In each case, one needs to define a surjective map from one of the group algebras to the Clifford algebra with kernel equal to the ideal $(1+\tau)$. Furthermore, one observes that for each $p+q \geq 0$, the number of non-isomorphic Salingaros groups of order 2^{p+q+1} equals the number of isomorphism classes of universal Clifford algebras $C\ell_{p,q}$ (see Periodicity of Eight in [13] and references therein).

Thus, for example, Salingaros groups in two classes N_3 and N_4 are sufficient to give two isomorphism classes of Clifford algebras of dimension sixteen: $C\ell_{0,4} \cong C\ell_{1,3} \cong C\ell_{4,0} \cong \mathbb{R}[N_4]/\mathcal{J}$, and $C\ell_{2,2} \cong C\ell_{3,1} \cong \mathbb{R}[N_3]/\mathcal{J}$.

This approach to the Periodicity of Eight of Clifford algebras would allow us to apply the representation theory of finite groups to explain some of the properties of Clifford algebras. For example, the fact that, up to an isomorphism, there are exactly two non-isomorphic non-Abelian groups of order eight, provides a group-theoretic explanation why there are exactly two isomorphism classes of Clifford algebras of dimension four. Furthermore, one observes that a construction of primitive idempotents [4, 13] needed for spinor representations of Clifford algebra $C\ell_{p,q}$ is possible due to the central-product structure of the related group $G_{p,q}$ in which the subgroups centralize each other and each subgroup contains at least one element of order 2.

This paper is organized as follows:

In Section 2 we provide three examples of Clifford algebras as projections of group algebras.

In Section 3 we discuss general properties of Salingaros vee groups $G_{p,q}$.

In Section 4, after providing necessary background material from group theory, we describe the central product structure of Salingaros vee groups. We recall Salingaros Theorem that classifies these groups into five isomorphism classes.

In Section 5, we state our Main Theorem. We prove that the group algebras $\mathbb{R}[G_{p,q}]$ are $\mathbb{Z}_2$-graded algebra. Then, we prove homogeneity of their ideals generated by $1+\tau$ for a central involution τ, and prove that the quotient algebras $\mathbb{R}[G_{p,q}]/\mathcal{J}$ are $\mathbb{Z}_2$-graded. We prove, using the group theory,

[1]By a small abuse of notation, we identify the group $G_{p,q}$ with its Salingaros class.

that the isomorphism $\mathbb{R}[Q_8]/\mathcal{J} \cong C\ell_{0,2}$ is $\mathbb{Z}_2$-graded while only one of two isomorphisms $\mathbb{R}[D_8]/\mathcal{J} \cong C\ell_{1,1}$ or $\mathbb{R}[D_8]/\mathcal{J} \cong C\ell_{2,0}$ is $\mathbb{Z}_2$-graded.

In Appendix B we show two algorithms to represent all Salingaros vee groups $G_{p,q}$ for $p+q \leq 9$ as internal central products of subgroups. These algorithms have been implemented in `CLIFFORD` [1, 2].

Throughout this paper, C_2 and C_4 are cyclic groups of order 2 and 4, respectively; $D_4 = C_2 \times C_2$; D_8 is the dihedral group of a square and Q_8 is the quaternion group, both of order 8. G', $Z(G)$, and $\Phi(G)$ denote the derived subgroup of G, the center of G, and Frattini subgroup of G, respectively. Notation $H \lhd G$ means that H is a normal subgroup of G while $[G : H]$ denotes the index of a subgroup H in G (defined as the number of left cosets of H in G) [11, 16].

2. Three examples of Clifford algebras as projections of group algebras

Let G be any finite group and let $\mathbb{F}$ be a field. Denote the group algebra of G over $\mathbb{F}$ as $\mathbb{F}[G]$, thus $\mathbb{F}[G] = \left\{\sum_{g \in G} \lambda_g g,\ \lambda_g \in \mathbb{F}\right\}$ with the algebra multiplication determined by the group product [11]. In this paper, we focus on real group algebras of finite 2-groups, in particular, Salingaros vee groups.

Definition 1. *Let p be a prime. A group G is a p-group if every element in G is of order p^k for some $k \geq 1$. So, any finite group G of order p^n is a p-group.*

There are exactly two non-abelian groups of order eight, namely, the quaternion group Q_8 and the dihedral group D_8. The quaternion group has two convenient presentations:

$$Q_8 = \langle a, b \mid a^4 = 1, a^2 = b^2, bab^{-1} = a^{-1} \rangle \tag{1a}$$

$$= \langle I, J, \tau \mid \tau^2 = 1, I^2 = J^2 = \tau, IJ = \tau JI \rangle \tag{1b}$$

with $I = a, J = b, \tau = a^2$, and $|a^2| = 2$, $|a| = |a^3| = |b| = |ab| = |a^2b| = |a^3b| = 4$. The order structure of Q_8 is $[1, 1, 6]$ [2] and its center $Z(Q_8) = \{1, a^2\} \cong C_2$. We present the dihedral group D_8 in two ways as well:

$$D_8 = \langle a, b \mid a^4 = b^2 = 1, bab^{-1} = a^{-1} \rangle \tag{2a}$$

$$= \langle \sigma, \tau \mid \sigma^4 = \tau^2 = 1, \tau\sigma\tau^{-1} = \sigma^{-1} \rangle \tag{2b}$$

with $\tau = a, \sigma = b$, so $|a^2| = |b| = |ab| = |a^2b| = |a^3b| = 2$, $|a| = |a^3| = 4$. The order structure of D_8 is $[1, 5, 2]$ and its center $Z(D_8) = \{1, a^2\} \cong C_2$. [3]

Recall the following two constructions of $\mathbb{H} = C\ell_{0,2}$ as $\mathbb{R}[Q_8]/\mathcal{J}$ and $C\ell_{1,1}$ as $\mathbb{R}[D_8]/\mathcal{J}$ due to Chernov [7].

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[2] This means that Q_8 has one element of order 1, one element of order 2, and six elements of order 4.

[3] It is well known [12, Lemma 2.1.9] that any dihedral group $D_{2^{n+1}}$ for $n \geq 2$ is a split extension of C_{2^n} by C_2 whereas any quaternion group $Q_{2^{n+1}}$ is a non-split extension of C_{2^n} by C_2. We do not discuss group extensions in this paper.

Example 1. *Define an* $\mathbb{R}$*-algebra map* ψ *from* $\mathbb{R}[Q_8] \to \mathbb{H} = \mathrm{sp}\{1, \mathbf{i}, \mathbf{j}, \mathbf{ij}\}$ *as:*

$$1 \mapsto 1, \quad \tau \mapsto -1, \quad I \mapsto \mathbf{i}, \quad J \mapsto \mathbf{j}, \tag{3}$$

Then, $\mathcal{J} = \ker \psi = (1+\tau)$ *for a central involution* $\tau = a^2$ *in* Q_8*, so* $\dim_{\mathbb{R}} \mathcal{J} = 4$ *and* ψ *is surjective. Let* $\pi : \mathbb{R}[Q_8] \to \mathbb{R}[Q_8]/\mathcal{J}$ *be the natural map* $u \mapsto u+\mathcal{J}$*. There exists an isomorphism* $\varphi : \mathbb{R}[Q_8]/\mathcal{J} \to \mathbb{H}$ *such that* $\varphi \circ \pi = \psi$ *and*

$$\pi(I^2) = I^2 + \mathcal{J} = \tau + \mathcal{J} \text{ and } \varphi(\pi(I^2)) = \psi(\tau) = -1 = (\psi(I))^2 = \mathbf{i}^2,$$
$$\pi(J^2) = J^2 + \mathcal{J} = \tau + \mathcal{J} \text{ and } \varphi(\pi(J^2)) = \psi(\tau) = -1 = (\psi(J))^2 = \mathbf{j}^2,$$
$$\pi(IJ + JI) = IJ + JI + \mathcal{J} = (1+\tau)JI + \mathcal{J} = \mathcal{J} \text{ and}$$
$$\varphi(\pi(IJ + JI)) = \psi(0) = 0 = \psi(I)\psi(J) + \psi(J)\psi(I) = \mathbf{ij} + \mathbf{ji}.$$

Thus, $\mathbb{R}[Q_8]/\mathcal{J} \cong \psi(\mathbb{R}[Q_8]) = C\ell_{0,2} \cong \mathbb{H}$ *provided the central involution* τ *is mapped into* -1*.*

Example 2. *Define an* $\mathbb{R}$*-algebra map* ψ *from* $\mathbb{R}[D_8] \to C\ell_{1,1}$ *as:*

$$1 \mapsto 1, \quad \tau \mapsto \mathbf{e}_1, \quad \sigma \mapsto \mathbf{e}_2, \tag{4}$$

where $C\ell_{1,1}$ *is generated by orthonormal generators* $\mathbf{e}_1, \mathbf{e}_2$ *satisfying relations* $(\mathbf{e}_1)^2 = 1, (\mathbf{e}_2)^2 = -1, \mathbf{e}_1\mathbf{e}_2 = -\mathbf{e}_2\mathbf{e}_1$*. Then,* $\ker \psi = (1+\sigma^2)$ *where* σ^2 *is a central involution* a^2 *in* D_8*. Let* $\mathcal{J} = (1+\sigma^2)$*. Thus,* $\dim_{\mathbb{R}} \mathcal{J} = 4$ *and* ψ *is surjective. Let* $\pi : \mathbb{R}[D_8] \to \mathbb{R}[D_8]/\mathcal{J}$ *be the natural map* $u \mapsto u + \mathcal{J}$*. There exists an isomorphism* $\varphi : \mathbb{R}[D_8]/\mathcal{J} \to C\ell_{1,1}$ *such that* $\varphi \circ \pi = \psi$ *and*

$$\pi(\tau^2) = \tau^2 + \mathcal{J} = 1 + \mathcal{J} \text{ and } \varphi(\pi(\tau^2)) = \psi(1) = \psi(\tau^2) = (\mathbf{e}_1)^2 = 1,$$
$$\pi(\sigma^2) = \sigma^2 + \mathcal{J} \text{ and } \varphi(\pi(\sigma^2)) = \psi(\sigma^2) = \psi(-1) = (\mathbf{e}_2)^2 = -1,$$
$$\pi(\tau\sigma + \sigma\tau) = \tau\sigma + \sigma\tau + \mathcal{J} = \sigma\tau(1+\sigma^2) + \mathcal{J} = \mathcal{J} \text{ and}$$
$$\varphi(\pi(\tau\sigma + \sigma\tau) = \psi(\tau)\psi(\sigma) + \psi(\sigma)\psi(\tau) = \psi(0) = \mathbf{e}_1\mathbf{e}_2 + \mathbf{e}_2\mathbf{e}_1 = 0.$$

Thus, $\mathbb{R}[D_8]/\mathcal{J} \cong C\ell_{1,1}$ *provided the central involution* σ^2 *is mapped into* -1*.*

Walley [21] extended Chernov's construction to $C\ell_{2,0}$ and represented it as $\mathbb{R}[D_8]/\mathcal{J}$ as shown in the following example.

Example 3. *Define an algebra map* ψ *from* $\mathbb{R}[D_8] \to C\ell_{2,0}$ *as:*

$$1 \mapsto 1, \quad \tau \mapsto \mathbf{e}_1, \quad \sigma \mapsto \mathbf{e}_1\mathbf{e}_2, \tag{5}$$

where $C\ell_{2,0}$ *is generated by orthonormal generators* $\mathbf{e}_1, \mathbf{e}_2$ *satisfying relations* $(\mathbf{e}_1)^2 = (\mathbf{e}_2)^2 = 1, \mathbf{e}_1\mathbf{e}_2 = -\mathbf{e}_2\mathbf{e}_1$*. Then,* $\ker \psi = (1+\sigma^2)$ *where* σ^2 *is a central involution* a^2 *in* D_8*. Let* $\mathcal{J} = (1+\sigma^2)$*. Thus,* $\dim_{\mathbb{R}} \mathcal{J} = 4$ *and* ψ *is surjective. Let* $\pi : \mathbb{R}[D_8] \to \mathbb{R}[D_8]/\mathcal{J}$ *be the natural map* $u \mapsto u + \mathcal{J}$*. There exists an isomorphism* $\varphi : \mathbb{R}[D_8]/\mathcal{J} \to C\ell_{2,0}$ *such that* $\varphi \circ \pi = \psi$ *and*

$$\pi(\tau^2) = \tau^2 + \mathcal{J} = 1 + \mathcal{J} \text{ and } \varphi(\pi(\tau^2)) = \psi(\tau^2) = \psi(1) = (\mathbf{e}_1)^2 = 1,$$
$$\pi(\sigma^2) = \sigma^2 + \mathcal{J} \text{ so } \varphi(\pi(\sigma^2)) = \psi(-1) = (\mathbf{e}_1\mathbf{e}_2)^2 = -1, \text{ so } (\mathbf{e}_2)^2 = 1 \text{ since}$$
$$\varphi(\pi(\tau\sigma + \sigma\tau)) = \varphi(\tau\sigma + \sigma\tau + \mathcal{J}) = \varphi(\sigma\tau(1+\sigma^2) + \mathcal{J}) = \varphi(\mathcal{J}) \text{ and}$$
$$\psi(\tau)\psi(\sigma) + \psi(\sigma)\psi(\tau) = \psi(0) = \mathbf{e}_1\mathbf{e}_1\mathbf{e}_2 + \mathbf{e}_1\mathbf{e}_2\mathbf{e}_1 = 0, \text{ so } \mathbf{e}_1\mathbf{e}_2 + \mathbf{e}_2\mathbf{e}_1 = 0.$$

Thus, $\mathbb{R}[D_8]/\mathcal{J} \cong C\ell_{2,0}$ provided the central involution σ^2 is mapped into -1.

Let us summarize the above three projective constructions. Notice first that each group $N_2 = Q_8$ and $N_1 = D_8$ can be written as follows:

1. The quaternion group Q_8:

$$Q_8 = \{\tau^{\alpha_0} g_1^{\alpha_1} g_2^{\alpha_2} \mid \alpha_k \in \{0,1\},\, k = 0,1,2\}$$

   where $\tau = a^2$ is the central involution in Q_8, $g_1 = a$, and $g_2 = b$. Thus,

$$(g_1)^2 = a^2 = \tau, \quad (g_2)^2 = b^2 = a^2 = \tau, \quad \tau g_1 g_2 = g_2 g_1.$$

   Observe that $|g_1| = |g_2| = 4$ and $\mathbb{R}[Q_8]/\mathcal{J} \cong C\ell_{0,2}$ where $\mathcal{J} = (1+\tau)$.

2. The dihedral group D_8:

$$D_8 = \{\tau^{\alpha_0} g_1^{\alpha_1} g_2^{\alpha_2} \mid \alpha_k \in \{0,1\},\, k = 0,1,2\}$$

   where $\tau = a^2$ is the central involution in D_8, $g_1 = b$, and $g_2 = a$. Thus,

$$(g_1)^2 = b^2 = 1, \quad (g_2)^2 = a^2 = \tau, \quad \tau g_1 g_2 = g_2 g_1.$$

   Observe that $|g_1| = 2$, $|g_2| = 4$ and $\mathbb{R}[D_8]/\mathcal{J} \cong C\ell_{1,1} \cong C\ell_{2,0}$ where $\mathcal{J} = (1+\tau)$.

Examples 1 and 2 illustrate Chernov Theorem [7]. Here is its reformulated version with a proof.

Theorem 1. *Let G be a finite 2-group of order 2^{1+n} generated by a central involution τ and additional elements $g_1, \ldots, g_n$, which satisfy the following relations:*

$$\tau^2 = 1, \quad (g_1)^2 = \cdots = (g_p)^2 = 1, \quad (g_{p+1})^2 = \cdots = (g_{p+q})^2 = \tau, \tag{6a}$$

$$\tau g_j = g_j \tau, \quad g_i g_j = \tau g_j g_i, \quad i,j = 1,\ldots,n = p+q, \tag{6b}$$

Let $\mathcal{J} = (1+\tau)$ be an ideal in the group algebra $\mathbb{R}[G]$ and let $C\ell_{p,q}$ be the universal real Clifford algebra generated by $\{\mathbf{e}_k\}, k = 1,\ldots,n = p+q$, where

$$\mathbf{e}_i^2 = Q(\mathbf{e}_i) \cdot 1 = \varepsilon_i \cdot 1 = \begin{cases} 1 & \textit{for } 1 \le i \le p; \\ -1 & \textit{for } p+1 \le i \le p+q; \end{cases} \tag{7a}$$

$$\mathbf{e}_i \mathbf{e}_j + \mathbf{e}_j \mathbf{e}_i = 0, \quad i \ne j, \quad 1 \le i,j \le n. \tag{7b}$$

Then, (a) $\dim_{\mathbb{R}} \mathcal{J} = 2^n$; (b) There exists a surjective $\mathbb{R}$-algebra homomorphism ψ from the group algebra $\mathbb{R}[G]$ to $C\ell_{p,q}$ with $\ker\psi = \mathcal{J}$.

Proof. Observe that $G = \{\tau^{\alpha_0} g_1^{\alpha_1} \cdots g_n^{\alpha_n}\} \mid \alpha_k \in \{0,1\},\, k = 0,1,\ldots,n\}$. The existence of a central involution τ is guaranteed by a well-known fact that the center of any p-group is nontrivial [16]. Define an $\mathbb{R}$-algebra homomorphism $\psi : \mathbb{R}[G] \to C\ell_{p,q}$ as:

$$1 \mapsto 1, \quad \tau \mapsto -1, \quad g_j \mapsto \mathbf{e}_j, \quad j = 1,\ldots,n. \tag{8}$$

Clearly, $\mathcal{J} \subset \ker\psi$. Let $u \in \mathbb{R}[G]$. Then, $u = \sum_{\alpha} \lambda_{\alpha} \tau^{\alpha_0} g_1^{\alpha_1} \cdots g_n^{\alpha_n} = u_1 + \tau u_2$ where $u_i = \sum_{\widetilde{\alpha}} \lambda_{\widetilde{\alpha}}^{(i)} g_1^{\alpha_1} \cdots g_n^{\alpha_n}$, $i = 1, 2$, $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_n) \in \mathbb{R}^{n+1}$ and $\widetilde{\alpha} = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$. Thus, if $u \in \ker\psi$, then

$$\psi(u) = \sum_{\widetilde{\alpha}} (\lambda_{\widetilde{\alpha}}^{(1)} - \lambda_{\widetilde{\alpha}}^{(2)}) \mathbf{e}_1^{\alpha_1} \cdots \mathbf{e}_n^{\alpha_n} = 0 \tag{9}$$

implies $\lambda_{\widetilde{\alpha}}^{(1)} = \lambda_{\widetilde{\alpha}}^{(2)}$ since $\{\mathbf{e}_1^{\alpha_1} \cdots \mathbf{e}_n^{\alpha_n} \mid (\alpha_1, \ldots, \alpha_n) \in (\mathbb{Z}_2)^n\}$ is a basis in $C\ell_{p,q}$. Hence,

$$u = (1 + \tau) \sum_{\widetilde{\alpha}} \lambda_{\widetilde{\alpha}}^{(1)} g_1^{\alpha_1} \cdots g_n^{\alpha_n} \in \mathcal{J}.$$

Thus, $\dim_{\mathbb{R}} \ker\psi = 2^n$, $\ker\psi = \mathcal{J}$, $\dim_{\mathbb{R}} \mathbb{R}[G]/\mathcal{J} = 2^{1+n} - 2^n = 2^n$, so ψ is surjective. Let $\varphi : \mathbb{R}[G]/\mathcal{J} \to C\ell_{p,q}$ be such that $\varphi \circ \pi = \psi$ where $\pi : \mathbb{R}[G] \to \mathbb{R}[G]/\mathcal{J}$ is the natural map. Then, since $\psi(g_j) = \mathbf{e}_j$, $\pi(g_j) = g_j + \mathcal{J}$, we have $\varphi(\pi(g_j)) = \varphi(g_j + \mathcal{J}) = \psi(g_j) = \mathbf{e}_j$ and

$$\begin{aligned} \pi(g_j)\pi(g_i) + \pi(g_i)\pi(g_j) &= (g_j + \mathcal{J})(g_i + \mathcal{J}) + (g_j + \mathcal{J})(g_i + \mathcal{J}) \\ &= (g_j g_i + g_i g_j) + \mathcal{J} = (1 + \tau) g_j g_i + \mathcal{J} = \mathcal{J} \end{aligned}$$

for $i \neq j$ since $g_i g_j = \tau g_j g_i$ in $\mathbb{R}[G]$, τ is central, and $\mathcal{J} = (1 + \tau)$. Thus, $g_j + \mathcal{J}, g_i + \mathcal{J}$ anticommute in $\mathbb{R}[G]/\mathcal{J}$ when $i \neq j$. Also, when $i = j$,

$$\pi(g_i)\pi(g_i) = (g_i + \mathcal{J})(g_i + \mathcal{J}) = (g_i)^2 + \mathcal{J} = \begin{cases} 1 + \mathcal{J}, & 1 \leq i \leq p; \\ \tau + \mathcal{J}, & p + 1 \leq i \leq n; \end{cases}$$

Observe, that $\tau + \mathcal{J} = (-1) + (1 + \tau) + \mathcal{J} = (-1) + \mathcal{J}$ in $\mathbb{R}[G]/\mathcal{J}$. We conclude that the factor algebra $\mathbb{R}[G]/\mathcal{J}$ is generated by the cosets $g_i + \mathcal{J}$ which satisfy these relations:

$$\begin{aligned} (g_j + \mathcal{J})(g_i + \mathcal{J}) + (g_j + \mathcal{J})(g_i + \mathcal{J}) &= \mathcal{J}, \\ (g_i)^2 + \mathcal{J} &= \begin{cases} 1 + \mathcal{J}, & 1 \leq i \leq p; \\ (-1) + \mathcal{J}, & p + 1 \leq i \leq n; \end{cases} \end{aligned}$$

Thus, the factor algebra $\mathbb{R}[G]/\mathcal{J}$ is isomorphic to $C\ell_{p,q}$ provided $\mathcal{J} = (1 + \tau)$ for the central involution τ in G. □

Note that Example 3 shows that the map $\psi : \mathbb{R}[G] \to C\ell_{p,q}$ need not be defined as in (8). This allows one to define different surjective maps ψ from the same group algebra $\mathbb{R}[G]$ to different but isomorphic Clifford algebras, e.g., $C\ell_{1,1} \cong C\ell_{2,0}$.

3. Salingaros vee groups $G_{p,q} \subset C\ell_{p,q}^*$

We begin by recalling a definition of a derived subgroup and its basic properties [8, 9, 12, 16].

Definition 2. *If G is a group and $x, y \in G$, then their commutator $[x, y]$ is the element $x^{-1}y^{-1}xy$. If X and Y are subgroups of G, then the commutator subgroup $[X, Y]$ of G is defined by $[X, Y] = \langle [x, y] \mid x \in X, y \in Y \rangle$. In particular, the derived subgroup G' of G is defined as $G' = [G, G]$.*

Proposition 1 ([16]). *Let G be a group.*

(i) *G' is a normal subgroup of G, and G/G' is abelian.*
(ii) *If $H \lhd G$ and G/H is abelian, then $G' \subseteq H$.*

Let $G_{p,q}$ be a finite group of order 2^{p+q+1} contained in $C\ell_{p,q}^*$ with a binary operation being the Clifford algebra product. $G_{p,q}$ can be presented as:

$$G_{p,q} = \langle -1, \mathbf{e}_1, \ldots, \mathbf{e}_n \mid [\mathbf{e}_i, \mathbf{e}_j] = -1 \text{ for } i \neq j, \\ [\mathbf{e}_i, -1] = 1, \ (-1)^2 = 1, \text{ and } \mathbf{e}_i^2 = \pm 1 \rangle, \quad (10)$$

where $[\cdot, \cdot]$ denotes the commutator of two group elements, $(\mathbf{e}_i)^2 = 1$ for $1 \leq i \leq p$ and $(\mathbf{e}_i)^2 = -1$ for $p + 1 \leq i \leq n = p + q$. In the following, the elements $\mathbf{e}_{\underline{i}} = \mathbf{e}_{i_1}\mathbf{e}_{i_2} \cdots \mathbf{e}_{i_k}$ will be denoted for short as $\mathbf{e}_{i_1 i_2 \cdots i_k}$ for $k \geq 1$ while $\mathbf{e}_\emptyset$ will be denoted as 1, the identity element of $G_{p,q}$ (and $C\ell_{p,q}^*$).

The groups $G_{p,q}$ are known as *Salingaros vee groups*. They have been classified by Salingaros [17–19] and later discussed by Varlamov [20], Helmstetter [10], Abłamowicz and Fauser [4, 5], Maduranga and Abłamowicz [15], and most recently by Brown [6] and Walley [21]. $G_{p,q}$ is a discrete subgroup of $\mathbf{Pin}(p, q) \subset \mathbf{\Gamma}_{p,q}$ where $\mathbf{\Gamma}_{p,q}$ is the Lipschitz group in $C\ell_{p,q}$ [13].

Let us recall a few basic facts about these groups.

1. $|G_{p,q}| = 2^{1+p+q}$, $|G'_{p,q}| = 2$ since $G'_{p,q} = \{\pm 1\}$.
2. $G_{p,q}$ is not simple because as a finite 2-group it has a normal subgroup of order 2^m for every $m \leq p + q + 1$. In particular, it has a subgroup of index 2.
3. The center $Z(G_{p,q})$ of $G_{p,q}$ is non-trivial since $2 \mid |Z(G_{p,q})|$ and so every group $G_{p,q}$ has a central element of order 2. It is well-known that for any prime p and a finite p-group $G \neq \{1\}$, the center of G is non-trivial [16].
4. Every element of $G_{p,q}$ is of order 1, 2, or 4. This follows directly from the definition of $G_{p,q}$ as well as from its central-product structure (to be discussed below).
5. Since $[G_{p,q} : G'_{p,q}] = |G_{p,q}|/|G'_{p,q}| = 2^{p+q}$, each $G_{p,q}$ has 2^{p+q} linear characters [11, 14].
6. The number N of conjugacy classes in $G_{p,q}$, hence, the number of irreducible inequivalent representations of $G_{p,q}$, is $1 + 2^{p+q}$ (resp. $2 + 2^{p+q}$) when $p + q$ is even (resp. odd) (cf. [14, 15]).
7. The center $Z(G_{p,q})$ is described by the following theorem (see also [20]):

Theorem 2. *Let $G_{p,q} \subset C\ell^*_{p,q}$. Then,*

$$Z(G_{p,q}) = \begin{cases} \{\pm 1\} \cong C_2 & \text{if } p-q \equiv 0,2,4,6 \pmod 8; \\ \{\pm 1, \pm\beta\} \cong C_2 \times C_2 & \text{if } p-q \equiv 1,5 \pmod 8; \\ \{\pm 1, \pm\beta\} \cong C_4 & \text{if } p-q \equiv 3,7 \pmod 8. \end{cases} \tag{11}$$

as a consequence of $Z(C\ell_{p,q}) = \mathrm{sp}\{1\}$ (resp. $\mathrm{sp}\{1,\beta\}$) when $p+q$ is even (resp. odd) where $\beta = \mathbf{e}_1\mathbf{e}_2\cdots\mathbf{e}_n$, $n = p+q$, is the unit pseudoscalar in $C\ell_{p,q}$.

8. Salingaros' classes N_{2k-1}, N_{2k}, Ω_{2k-1}, Ω_{2k}, and S_k [17–19] are related to the groups $G_{p,q}$ as follows:

$$\begin{aligned} N_{2k-1} &\leftrightarrow G_{p,q} \subset C\ell^*_{p,q},\ p-q \equiv 0,2 \pmod 8, \quad \mathbb{K} \cong \mathbb{R}; \\ N_{2k} &\leftrightarrow G_{p,q} \subset C\ell^*_{p,q},\ p-q \equiv 4,6 \pmod 8, \quad \mathbb{K} \cong \mathbb{H}; \\ \Omega_{2k-1} &\leftrightarrow G_{p,q} \subset C\ell^*_{p,q},\ p-q \equiv 1 \pmod 8, \quad \mathbb{K} \cong \mathbb{R} \oplus \mathbb{R}; \\ \Omega_{2k} &\leftrightarrow G_{p,q} \subset C\ell^*_{p,q},\ p-q \equiv 5 \pmod 8, \quad \mathbb{K} \cong \mathbb{H} \oplus \mathbb{H}; \\ S_k &\leftrightarrow G_{p,q} \subset C\ell^*_{p,q},\ p-q \equiv 3,7 \pmod 8, \quad \mathbb{K} \cong \mathbb{C}. \end{aligned} \tag{12}$$

Table 1. Five isomorphism classes of vee groups $G_{p,q}$ in $C\ell^*_{p,q}$

Group G	$Z(G)$	Group order	$\dim C\ell_{p,q}$	$Z(C\ell_{p,q})$
N_{2k-1}	C_2	2^{2k+1}	2^{2k}	$\mathrm{sp}\{1\}$
N_{2k}	C_2	2^{2k+1}	2^{2k}	$\mathrm{sp}\{1\}$
Ω_{2k-1}	$C_2 \times C_2$	2^{2k+2}	2^{2k+1}	$\mathrm{sp}\{1,\beta\}$
Ω_{2k}	$C_2 \times C_2$	2^{2k+2}	2^{2k+1}	$\mathrm{sp}\{1,\beta\}$
S_k	C_4	2^{2k+2}	2^{2k+1}	$\mathrm{sp}\{1,\beta\}$

The first few vee groups $G_{p,q}$ corresponding to Clifford algebras $C\ell_{p,q}$ in dimensions $n = p+q = 1,2,3$, are:

$$\begin{aligned} n=1:&\quad G_{0,1} \cong S_0 \cong C_4, \quad G_{1,0} \cong \Omega_0 \cong D_4, \\ n=2:&\quad G_{0,2} \cong N_2 \cong Q_8, \quad G_{1,1} \cong N_1 \cong D_8, \quad G_{2,0} \cong N_1 \cong D_8, \\ n=3:&\quad G_{0,3} \cong \Omega_2, \quad G_{1,2} \cong S_1, \quad G_{2,1} \cong \Omega_1, \quad G_{3,0} \cong S_1. \end{aligned}$$

See Table 2 for all groups for $n \le 8$.

Table 2. Isomorphism classes of groups $G_{p,q}$ for $n = p+q \leq 8$.

n \ p-q	−8	−7	−6	−5	−4	−3	−2	−1	0	1	2	3	4	5	6	7	8
0									N_0								
1								S_0		Ω_0							
2							N_2		N_1		N_1						
3						Ω_2		S_1		Ω_1		S_1					
4					N_4		N_4		N_3		N_3		N_4				
5				S_2		Ω_4		S_2		Ω_3		S_2		Ω_4			
6			N_5		N_6		N_6		N_5		N_5		N_6		N_6		
7		Ω_5		S_3		Ω_6		S_3		Ω_5		S_3		Ω_6		S_3	
8	N_7		N_7		N_8		N_8		N_7		N_7		N_8		N_8		N_7

4. Central product structure of $G_{p,q}$

In this section we review basic results on extra-special p-groups from [8,9,12] that lead to Salingaros Theorem for the groups $G_{p,q}$.

Definition 3 ([9]). *A finite abelian p-group is elementary abelian if every non-trivial element has order p.*

For example, $(C_p)^k = C_p \times \cdots \times C_p$ (k-times) is elementary abelian. In particular, $D_4 = C_2 \times C_2$ and C_2 are elementary abelian.

We adopt the following definition of extra-special p-group. [4]

Definition 4 ([8]). *A finite p-group P is extra-special if*

(i) $Z(P) = P'$,
(ii) $|Z(P)| = |P'| = p$, *and*
(iii) P/P' *is elementary abelian.*

The groups D_8 and Q_8 are extra-special and non-isomorphic.

Example 4 (D_8 is extra-special). $D_8 = \langle a, b \mid a^4 = b^2 = 1, bab^{-1} = a^{-1}\rangle$ *is extra-special because:*

(i) $Z(D_8) = D_8' = [D_8, D_8] = \langle a^2 \rangle$, $|Z(D_8)| = |D_8'| = 2$,
(ii) $D_8/D_8' = D_8/Z(D_8) = \langle D_8', aD_8', bD_8', abD_8' \rangle \cong C_2 \times C_2$.
(iii) *Order structure:* $[1, 5, 2]$
(iv) $D_8 \cong C_4 \rtimes C_2 \cong (C_2 \times C_2) \rtimes C_2$ *(semi-direct products)*

Example 5 (Q_8 is extra-special). $Q_8 = \langle a, b \mid a^4 = 1, a^2 = b^2, bab^{-1} = a^{-1}\rangle$ *is extra-special because:*

(i) $Z(Q_8) = Q_8' = [Q_8, Q_8] = \langle a^2 \rangle$, $|Z(Q_8)| = |Q_8'| = 2$,
(ii) $Q_8/Q_8' = Q_8/Z(Q_8) = \langle Q_8', aQ_8', bQ_8', abQ_8' \rangle \cong C_2 \times C_2$.
(iii) *Order structure:* $[1, 1, 6]$
(iv) Q_8 *is not a semi-direct product of any of its subgroups (cf. Brown [6])*

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[4] An equivalent definition of extra-special p-group is as follows: *An extra-special group is a finite p-group G such that $Z(G) = G' = \Phi(G)$ is of order p.* Here, $\Phi(G)$ is Frattini subgroup of G defined as an intersection of all the maximal subgroups of G (cf. [12]).

We adopt the following definition from [8] of a group being a central product of its subgroups. However, some authors define the central product differently (cf. [9, 12]).

Definition 5 ([8]). *A finite group G is a central product of subgroups $H_1, \dots, H_n$ if $G = H_1 \cdots H_n$ and, for any $x \in H_i$, $y \in H_j$, $i \neq j$ implies $[x, y] = 1$. Then, we will write $G = H_1 \circ \cdots \circ H_n$. In particular, when $n = 2$, we have: (a) $[H, K] = \langle 1 \rangle$ and (b) $G = HK$.*

Note that the central product is commutative and associative, and when $G = H_1 \circ \cdots \circ H_n$, we have $\cap_{i=1}^{n} H_i \leq Z(G)$, $H_i \lhd G$, and $Z(H_i) \leq Z(G)$ for each i.

The following group-theoretic results provide a foundation for Salingaros Theorem. Here, p is a prime.

Lemma 1 ([8]). *Let $P_1, \dots, P_n$ be extra-special p-groups of order p^3. Then there is one and up to isomorphism only one central product of $P_1, \dots, P_n$ with center of order p. It is extra special of order p^{2n+1} denoted by $P_1 \circ \cdots \circ P_n$, and called the central product of $P_1, \dots, P_n$.*

Remark 1. *An important consequence of Lemma 1 is the following observation: Suppose that H and K are extra-special groups of order p^3 and G is their internal central product. Then, $Z(H) = Z(K) = Z(G)$. In particular, let $p = 2$ and suppose $G = D_8^{(2)} \circ D_8^{(1)}$ or $G = D_8^{(2)} \circ Q_8$ where the dihedral and the quaternion groups have the standard presentations as follows [5]:*

$$D_8^{(i)} = \langle a_i, b_i \mid (a_i)^4 = (b_i)^2 = 1, (b_i)^{-1} a_i b_i = (a_i)^{-1} \rangle, \quad i = 1, 2;$$
$$Q_8 = \langle a_3, b_3 \mid (a_3)^4 = 1, (a_3)^2 = (b_3)^2, (b_3)^{-1} a_3 b_3 = (a_3)^{-1} \rangle.$$

Since $Z(D_8^{(i)}) = \{1, (a_i)^2\}$, $i = 1, 2$, and $Z(Q_8) = \{1, (a_3)^2\}$, the equality of the centers $Z(D_8^{(1)}) = Z(D_8^{(2)}) = Z(Q_8)$ implies that $(a_1)^2 = (a_2)^2 = (a_3)^2$.

Lemma 2 ([8]). *The groups $Q_8 \circ Q_8$ and $D_8 \circ D_8$ are isomorphic of order 32, not isomorphic to $D_8 \circ Q_8$. If C is a cyclic 2-group of order at least 4, then $C \circ Q_8 \cong C \circ D_8$.*

The following two results tell us that for any prime p, there are exactly two isomorphism classes of extra-special p-groups.

Lemma 3 ([12]). *An extra-special p-group has order p^{2n+1} for some positive integer n, and is the iterated central product of non-abelian groups of order p^3.*

Theorem 3 ([12]). *There are exactly two isomorphism classes of extra-special groups of order 2^{2n+1} for positive integer n. One isomorphism type arises as the iterated central product of n copies of D_8; the other as the iterated central product of n groups isomorphic to D_8 and Q_8, including at least one copy of Q_8. That is, $D_8 \circ D_8 \circ \cdots \circ D_8$, or, $D_8 \circ D_8 \circ \cdots \circ D_8 \circ Q_8$, where it is understood that these are iterated central products.*

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[5]By $D_8^{(i)}$, $i = 1, 2, \dots$, we denote different yet isomorphic copies of D_8.

In Appendix B we show how to decompose the vee groups $G_{p,q}$ into internal central products using this iterative approach.

Theorem 4 (Salingaros Theorem [19]). *Let $N_1 = D_8$, $N_2 = Q_8$, and $(G)^{\circ k}$ be the iterated central product $G \circ \cdots \circ G$ (k times) of G. Then, for $k \geq 1$:*

1. $N_{2k-1} \cong (N_1)^{\circ k} = (D_8)^{\circ k}$,
2. $N_{2k} \cong (N_1)^{\circ k} \circ N_2 = (D_8)^{\circ(k-1)} \circ Q_8$,
3. $\Omega_{2k-1} \cong N_{2k-1} \circ (C_2 \times C_2) = (D_8)^{\circ k} \circ (C_2 \times C_2)$,
4. $\Omega_{2k} \cong N_{2k} \circ (C_2 \times C_2) = (D_8)^{\circ(k-1)} \circ Q_8 \circ (C_2 \times C_2)$,
5. $S_k \cong N_{2k-1} \circ C_4 \cong N_{2k} \circ C_4 = (D_8)^{\circ k} \circ C_4 \cong (D_8)^{\circ(k-1)} \circ Q_8 \circ C_4$.

Thus, the families N_{2k-1} and N_{2k} are the two isomorphism classes of extra-special groups of order 2^{2k+1} predicted by Theorem 3 while the three classes $\Omega_{2k-1}, \Omega_{2k}, S_k$ contain non extra-special groups of order 2^{2k+2}.

As a follow up to Remark 1, we make the following observations.

Remark 2. *Let $G = G_{p,q}$.*

(i) *When $G \cong N_{2k-1}$ or $G \cong N_{2k}$, the centers of the dihedral and the quaternion groups in the central product factorization of G are equal to $Z(G)$ with order 2. This implies that the squares of all their generators a_i of order 4 equal τ, the single central involution in G which in turn is just -1.*

(ii) *When $G \cong \Omega_{2k-1}$ or $G \cong \Omega_{2k}$, then $Z(G) \cong C_2 \times C_2$. Thus, from the Product Formula [16],*

$$|G| = 2^{2k+2} = \frac{|N_{2k-1}||C_2 \times C_2|}{|N_{2k-1} \cap (C_2 \times C_2)|} = \frac{2^{2k+3}}{|N_{2k-1} \cap (C_2 \times C_2)|},$$

we find that $|N_{2k-1} \cap (C_2 \times C_2)| = 2$. Since

$$Z(N_{2k-1}) \lneq Z(C_2 \times C_2) = C_2 \times C_2,$$

(and similarly for $Z(N_{2k})$) we conclude that the squares of the generators a_i must equal one of three central involutions in $C_2 \times C_2 \cong \{1, \beta, -1, -\beta\}$ where β is the unit pseudoscalar in $G_{p,q} \subset C\ell^*_{p,q}$ of order 2. However, since each generator a_i is a basis monomial in $C\ell_{p,q}$, its square is invariant under the grade involution automorphism α of $C\ell_{p,q}$, that is, $\alpha((a_i)^2) = (a_i)^2$, yet, $\alpha(\pm\beta) = \mp\beta$ since $p+q$ is odd. Like in (i), we conclude that the squares of all the generators a_i of order 4 equal $\tau = -1$, the central involution in G.

(iii) *When $G \cong S_k$, we recall that $Z(G) \cong C_4 \cong \langle\beta\rangle$ and so $Z(G)$ contains exactly one element of order 2, namely, $\beta^2 = -1$. Thus, $Z(N_{2k-1}) \cap Z(G) \cong Z(N_{2k}) \cap Z(G) \cong C_2$, and again, the squares of all the generators a_i of order 4 equal $\tau = -1$.*

As an example of application of Algorithms 1 and 2 presented in Appendix B, we show how the groups $G_{p,q}$ of order 16 can be written as internal central products.

Example 6. *Let $p+q=3$. The central-product factorizations of groups $G_{p,q}$ may be given as follows:*

$$G_{0,3} \cong \Omega_2 \cong N_2 \circ (C_2 \times C_2) \cong \langle \mathbf{e}_1, \mathbf{e}_2 \rangle \circ \langle \beta, -\beta \rangle,$$
$$G_{1,2} \cong S_1 \cong N_1 \circ C_4 \cong \langle \mathbf{e}_2, \mathbf{e}_1 \rangle \circ \langle \beta \rangle \cong N_2 \circ C_4 \cong \langle \mathbf{e}_2, \mathbf{e}_3 \rangle \circ \langle \beta \rangle,$$
$$G_{2,1} \cong \Omega_1 \cong N_1 \circ (C_2 \times C_2) \cong \langle \mathbf{e}_3, \mathbf{e}_1 \rangle \circ \langle \beta, -\beta \rangle,$$
$$G_{3,0} \cong S_1 \cong N_1 \circ C_4 \cong \langle \mathbf{e}_{12}, \mathbf{e}_1 \rangle \circ \langle \beta \rangle \cong N_2 \circ C_4 \cong \langle \mathbf{e}_{12}, \mathbf{e}_3 \rangle \circ \langle \beta \rangle,$$

where β is a unit pseudoscalar, $N_1 = D_8$, $N_2 = Q_8$, and we have shortened our notation to only show generators for all the subgroups [6].

5. Clifford algebras $C\ell_{p,q}$ as images of group algebras $\mathbb{R}[G_{p,q}]$

Since the relations defining $G_{p,q}$ shown in (10) are the same as relations (6) assumed in Theorem 1 (with -1 being a central involution), we have the following result.

Main Theorem. *Every Clifford algebra $C\ell_{p,q}$, $p+q \geq 2$, is $\mathbb{R}$-isomorphic to a quotient of the group algebra $\mathbb{R}[G_{p,q}]$ of Salingaros vee group $G_{p,q}$ of order 2^{p+q+1} modulo an ideal $\mathcal{J} = (1+\tau)$ generated by $1+\tau$ for a central element τ of order 2.*

The above theorem allows us to classify Clifford algebras through the groups $G_{p,q}$. Notice from Table 2 that for each $n = p+q$, there is a bijective correspondence between the isomorphism classes of the groups $G_{p,q}$, hence, the group algebras $\mathbb{R}[G_{p,q}]$ and their quotients, and the isomorphism classes of Clifford algebras $C\ell_{p,q}$ (check Periodicity of Eight in [13, Table 1, p. 217]). For example, for $n=4$, we have two isomorphism classes:

$$\mathbb{R}[N_4]/\mathcal{J} \cong C\ell_{0,4} \cong C\ell_{1,3} \cong C\ell_{4,0} \quad \text{and} \quad \mathbb{R}[N_3]/\mathcal{J} \cong C\ell_{2,2} \cong C\ell_{3,1}.$$

5.1. $\mathbb{Z}_2$-gradation of $\mathbb{R}[G_{p,q}]$

Recall the following well-known theorem in group theory.

Proposition 2 ([16]). *If G is a p-group of order p^n, then G has a normal subgroup of order p^k for every $k \leq n$.*

The following result is an immediate consequence of this proposition.

Proposition 3. *Let G be Salingaros vee group $G_{p,q}$. Then,*

(i) *G has a normal subgroup H of index 2.*
(ii) *$G = H \dot\cup Hb$ for some element $b \notin H$ such that $b^2 \in H$.*
(iii) *The group algebra $\mathbb{R}[G]$ is $\mathbb{Z}_2$-graded.*

Proof of (i): Follows from Proposition 2.

[6] For example, $N_2 = Q_8 \cong \langle \mathbf{e}_1, \mathbf{e}_2 \rangle$ means that $a = \mathbf{e}_1$, $b = \mathbf{e}_2$, $a^4 = 1$, $a^2 = b^2$, and $b^{-1}ab = a^{-1}$ in agreement with (1a).

Proof of (ii): Since $[G : H] = 2$, we have $G/H \cong C_2$ and so $G = H \dot\cup Hb$ where $b \notin H$ yet $b^2 \in H$.

Proof of (iii): Since $G = H \dot\cup Hb$, we have

$$\mathbb{R}[G] = \left\{ \sum_{h \in H} x_h h + \sum_{h \in H} y_h hb \mid x_h, y_h \in \mathbb{R} \right\}.$$

Let

$$(\mathbb{R}[G])^{(0)} = \left\{ \sum_{h \in H} x_h h \mid x_h \in \mathbb{R} \right\}, \quad (\mathbb{R}[G])^{(1)} = \left\{ \sum_{h \in H} y_h hb \mid y_h \in \mathbb{R} \right\}. \quad (13)$$

Then, since $H \lhd G$, $b \notin H$, and $b^2 \in H$, we have

$$\mathbb{R}[G] = (\mathbb{R}[G])^{(0)} \oplus (\mathbb{R}[G])^{(1)},$$
$$(\mathbb{R}[G])^{(i)} (\mathbb{R}[G])^{(j)} \subseteq (\mathbb{R}[G])^{(i+j \bmod 2)}, \quad i, j = 0, 1. \qquad \square$$

For the given group $G_{p,q}$, there are usually several choices for the maximal subgroup H in the above proposition. In the following we will show that our results do not depend on that choice.

5.2. Homogeneity of the ideal $\mathcal{J} = (1 + \tau)$

We recall the following result from group theory (cf. [11, 16]).

Lemma 4. *Let G be a group of order p^n with $n \geq 1$. If $\{1\} \neq H \lhd G$ then $H \cap Z(G) \neq \{1\}$. In particular, $Z(G) \neq \{1\}$.*

When $G = G_{p,q}$ and $\{1\} \neq H \lhd G$, we have $2 \mid |H \cap Z(G_{p,q})|$. In particular, when $[G : H] = 2$, the central involution $\tau \in H$. In fact, in that case, $|H \cap Z(G_{p,q})| = 2$.

Lemma 5. *Let $G = G_{p,q}$, $p + q \geq 2$,* [7] *and let $H \lhd G$ such that $[G : H] = 2$.*

- (i) *If the isomorphism class of G is N_{2k-1} or N_{2k}, and τ is the central involution in $Z(G)$, then $\tau \in H$.*
- (ii) *If the isomorphism class of G is Ω_{2k-1} or Ω_{2k}, then there exits a central involution τ in $Z(G)$ which is contained in H.*
- (ii) *If the isomorphism class of G is S_k, and τ is the central involution in $Z(G)$, then $\tau \in H$.*

Proof of (i): Since $H \lhd G$ and $[G : H] = 2$, the quotient group G/H is abelian and so by part (b) of Proposition 1, $G' = \{1, -1\} \subset H$. Thus, $\tau = -1 \in H$.

Proof of (ii): Since $Z(G) \cong C_2 \times C_2$, the center of G contains three elements of order 2. Note that since

$$\Omega_{2k-1} \cong N_{2k-1} \circ (C_2 \times C_2) \quad \text{and} \quad \Omega_{2k} \cong N_{2k} \circ (C_2 \times C_2),$$

[7] For the special case when $p + q = 1$, see Appendix A.

and $[\Omega_{2k-1} : N_{2k-1}] = [\Omega_{2k} : N_{2k}] = 2$, from the Product Formula we find that $|N_{2k-1} \cap Z(G)| = |N_{2k} \cap Z(G)| = 2$. Thus, one of the central involutions, call it τ, belongs to $Z(N_{2k-1})$ or $Z(N_{2k})$, respectively, and so it must again equal a^2 where a is a generator of order 4 of any subgroup N_1 or N_2. If H is any subgroup in $G_{p,q}$ of index 2, it must contain an element of order 4, hence, it will contain the central element -1. Thus, again, $\tau = a^2 = -1$.

Proof of (iii): In this case, since $Z(G) \cong C_4$, there is a single central involution τ in $Z(G)$. Thus, $\tau \in H$. We can also observe from $S_k \cong N_{2k-1} \circ C_4 \cong N_{2k} \circ C_4$ that the normal subgroups N_{2k-1} and N_{2k} are of index 2. Thus, $\tau = a^2 = -1$ as well. ☐

Proposition 4. *Let $G = G_{p,q}$. Ideal $\mathcal{J} = (1+\tau)$ in $\mathbb{R}[G]$ is homogeneous.*

Proof. Let $J = (1+\tau) \subset \mathbb{R}[G] = (\mathbb{R}[G])^{(0)} \oplus (\mathbb{R}[G])^{(1)}$ where $\tau = a^2 \in Z(G)$. Let H be as in Proposition 3. Then, $\tau \in (\mathbb{R}[G])^{(0)}$ and the ideal $\mathcal{J}$ is homogeneous. ☐

As a consequence of being homogeneous, the ideal $\mathcal{J}$, as a subalgebra of $\mathbb{R}[G]$, is also $\mathbb{Z}_2$-graded. Let $j = u(1+\tau) \in \mathcal{J}$ where $u \in \mathbb{R}[G]$. Then, $\mathcal{J}$ is homogeneous because the homogeneous parts of j belong to $\mathcal{J}$:

$$\mathcal{J} \ni j = u(1+\tau) = \underbrace{u^{(0)}(1+\tau)}_{j^{(0)} \in \mathcal{J}^{(0)}} + \underbrace{u^{(1)}(1+\tau)}_{j^{(1)} \in \mathcal{J}^{(1)}} \tag{14}$$

where $u^{(i)} \in (\mathbb{R}[G])^{(i)}$, $\mathcal{J}^{(i)} = (1+\tau)(\mathbb{R}[G])^{(i)}$. Clearly, $\mathcal{J} = \mathcal{J}^{(0)} \oplus \mathcal{J}^{(1)}$ and $\mathcal{J}^{(i)}\mathcal{J}^{(j)} \subseteq \mathcal{J}^{(i+j \bmod 2)}$, $i, j = 0, 1$.

5.3. $\mathbb{Z}_2$-gradation of $\mathbb{R}[G_{p,q}]/\mathcal{J}$

We have the following result.

Proposition 5. *Let $G = G_{p,q}$ and $\mathcal{J} = (1+\tau)$ be the homogeneous ideal in $\mathbb{R}[G]$ where τ is a central involution in G. Let H be normal in G of index 2, $b \in G \setminus H, b^2 \in H$. Then, the quotient algebra $\mathbb{R}[G]/\mathcal{J}$ is $\mathbb{Z}_2$-graded:*

$$\mathbb{R}[G]/\mathcal{J} = (\mathbb{R}[G]/\mathcal{J})^{(0)} \oplus (\mathbb{R}[G]/\mathcal{J})^{(1)} \quad \textit{with} \tag{15a}$$

$$(\mathbb{R}[G]/\mathcal{J})^{(0)} = \mathrm{sp}\{h + \mathcal{J} \mid h \in H\}, \quad \textit{and} \tag{15b}$$

$$(\mathbb{R}[G]/\mathcal{J})^{(1)} = \mathrm{sp}\{hb + \mathcal{J} \mid h \in H\}. \tag{15c}$$

Proof. From Proposition 3, we know that $\mathbb{R}[G]$ is $\mathbb{Z}_2$-graded. Then, if we define $(\mathbb{R}[G]/\mathcal{J})^{(i)}$ as in (15b) and (15c), then clearly (15a) holds. Furthermore,

$$(h+\mathcal{J})(h'+\mathcal{J}) = hh' + \mathcal{J} \in (\mathbb{R}[G]/\mathcal{J})^{(0)}, \tag{16a}$$

$$(hb+\mathcal{J})(h'b+\mathcal{J}) = hbh'b + \mathcal{J} = h(bh'b^{-1})b^2 + \mathcal{J} \in (\mathbb{R}[G]/\mathcal{J})^{(0)}, \tag{16b}$$

$$(h+\mathcal{J})(h'b+\mathcal{J}) = hh'b + \mathcal{J} \in (\mathbb{R}[G]/\mathcal{J})^{(1)}, \tag{16c}$$

$$(h'b+\mathcal{J})(h+\mathcal{J}) = h'bh + \mathcal{J} = h'(bhb^{-1})b + \mathcal{J} \in (\mathbb{R}[G]/\mathcal{J})^{(1)}. \tag{16d}$$

Thus, $(\mathbb{R}[G]/\mathcal{J})^{(i)} (\mathbb{R}[G]/\mathcal{J})^{(j)} \subseteq (\mathbb{R}[G]/\mathcal{J})^{(i+j \bmod 2)}$, $i, j = 0, 1$. ☐

Example 7. *Consider the group algebra* $\mathbb{R}[Q_8]$ *from Example* 1. *Then,*

$$\mathbb{R}[Q_8] = (\mathbb{R}[Q_8])^{(0)} \oplus (\mathbb{R}[Q_8])^{(1)} \tag{17}$$

where $(\mathbb{R}[Q_8])^{(0)} = \mathrm{sp}\{1, I, \tau, \tau I\}$ *and* $(\mathbb{R}[Q_8])^{(1)} = \mathrm{sp}\{J, IJ, \tau J, \tau IJ\}$. *Furthermore,* $\mathcal{J} = \mathcal{J}^{(0)} \oplus \mathcal{J}^{(1)}$ *where* $\mathcal{J}^{(0)} = \mathrm{sp}\{1+\tau, (1+\tau)I\}$ *and* $\mathcal{J}^{(1)} = \mathrm{sp}\{(1+\tau)J, (1+\tau)IJ\}$. *The quotient algebra decomposes as*

$$\mathbb{R}[Q_8]/\mathcal{J} = (\mathbb{R}[Q_8]/\mathcal{J})^{(0)} \oplus (\mathbb{R}[Q_8]/\mathcal{J})^{(1)} \tag{18}$$

where

$$(\mathbb{R}[Q_8]/\mathcal{J})^{(0)} = \mathrm{sp}\{1+\mathcal{J}, IJ+\mathcal{J}\}, \tag{19}$$

$$(\mathbb{R}[Q_8]/\mathcal{J})^{(1)} = \mathrm{sp}\{I+\mathcal{J}, J+\mathcal{J}\}. \tag{20}$$

It can also be verified that the following $\mathbb{R}$*-algebra map*

$$\mathbb{R}[Q_8]/\mathcal{J} = (\mathbb{R}[Q_8]/\mathcal{J})^{(0)} \oplus (\mathbb{R}[Q_8]/\mathcal{J})^{(1)} \xrightarrow{\varphi} (\mathbb{R}[Q_8])^{(0)}/\mathcal{J}^{(0)} \oplus (\mathbb{R}[Q_8])^{(1)}/\mathcal{J}^{(1)} \tag{21}$$

defined as

$$1+\mathcal{J} \mapsto 1+\mathcal{J}^{(0)}, \qquad 1+\mathcal{J} \mapsto 1+\mathcal{J}^{(0)}, \tag{22a}$$

$$I+\mathcal{J} \mapsto J+\mathcal{J}^{(1)}, \qquad J+\mathcal{J} \mapsto IJ+\mathcal{J}^{(1)} \tag{22b}$$

is a $\mathbb{Z}_2$*-graded isomorphism. Here, the codomain of* φ *is a* $\mathbb{Z}_2$*-graded algebra with multiplication defined as*

$$(a+\mathcal{J}^{(i)})(b+\mathcal{J}^{(j)}) = ab + \mathcal{J}^{(i+j \bmod 2)}, \quad i,j = 0,1. \tag{23}$$

See Appendix C *for multiplication tables in* $\mathbb{R}[Q_8]/\mathcal{J}$ *and* $(\mathbb{R}[Q_8])^{(0)}/\mathcal{J}^{(0)} \oplus (\mathbb{R}[Q_8])^{(1)}/\mathcal{J}^{(1)}$.

5.4. On the $\mathbb{Z}_2$-graded isomorphism $\mathbb{R}[G_{p,q}]/\mathcal{J} \cong C\ell_{p,q}$

Our Main Theorem in Section 5 states that each Clifford algebra $C\ell_{p,q}$ is $\mathbb{R}$-isomorphic to the quotient algebra $\mathbb{R}[G_{p,q}]/\mathcal{J}$ where $G_{p,q}$ is Salingaros vee group contained in $C\ell^*_{p,q}$. However, both algebras are $\mathbb{Z}_2$-graded. Thus, a natural question is to ask whether the $\mathbb{R}$-isomorphism $\mathbb{R}[G_{p,q}]/\mathcal{J} \cong C\ell_{p,q}$ is also $\mathbb{Z}_2$-graded (or, just graded, for short). Of course, not all such isomorphisms may be graded. This is because Clifford algebras belonging to the same isomorphism class –as predicted by the Periodicity of Eight Theorem– are only $\mathbb{R}$-isomorphic. For example, since the two algebras $C\ell_{1,1}$ and $C\ell_{2,0}$ are $\mathbb{R}$-isomorphic to $\mathrm{Mat}(2,\mathbb{R})$, they are not graded-isomorphic. Hence, for one, we cannot expect both isomorphisms $\mathbb{R}[G_{1,1}]/\mathcal{J} \cong C\ell_{1,1}$ and $\mathbb{R}[G_{2,0}]/\mathcal{J} \cong C\ell_{2,0}$ to be simultaneously graded. For two, when the isomorphism $\mathbb{R}[G_{p,q}]/\mathcal{J} \cong C\ell_{p,q}$ is graded, we would expect it to remain graded regardless which maximal subgroup H of $G_{p,q}$ has been used to exhibit the $\mathbb{Z}_2$-gradation of the group algebra $\mathbb{R}[G_{p,q}]$, the ideal $\mathcal{J}$, and, in the end, the quotient algebra $\mathbb{R}[G_{p,q}]/\mathcal{J}$.

Our goal for this section is to investigate these issues on two examples. In the end, we will conjecture a general result.

5.5. Is the isomorphism $\mathbb{R}[Q_8]/\mathcal{J} \cong C\ell_{0,2}$ graded?

Recall presentation (1a) of Q_8 and a multiplication Table 18 of Q_8 in Appendix D. Q_8 has three maximal subgroups, each isomorphic to C_4:

$$H_1 = \langle a \rangle = \{1, \tau, a, \tau a\},\ H_2 = \langle b \rangle = \{1, b, \tau, \tau b\},\ H_3 = \langle ab \rangle = \{1, \tau, ab, \tau ab\}.$$

Each subgroup is of index 2, hence it is normal, and their intersection, as expected, is $\Phi(Q_8) = Q_8' = Z(Q_8) = \{1, \tau\}$ since Q_8 is extra-special. Let H be any of these three subgroups. In the following, we will base the graded structure of $\mathbb{R}[Q_8]$, $\mathcal{J}$ and $\mathbb{R}[Q_8]/\mathcal{J}$ on H, and show that the isomorphism $\mathbb{R}[Q_8]/\mathcal{J} \cong C\ell_{0,2}$ is graded.

5.5.1. Case $H = H_1$: Since $Q_8 = H_1 \dot\cup H_1 b = \{1, \tau, a, \tau a\} \dot\cup \{b, ab, \tau b \tau ab\}$,

$$\mathbb{R}[Q_8] = \underbrace{\mathrm{sp}\{1, \tau, a, \tau a\}}_{(\mathbb{R}[Q_8])^{(0)}} \oplus \underbrace{\mathrm{sp}\{b, ab, \tau b, \tau ab\}}_{(\mathbb{R}[Q_8])^{(1)}} \tag{24}$$

and $(\mathbb{R}[Q_8])^{(i)}(\mathbb{R}[Q_8])^{(j)} \subseteq (\mathbb{R}[Q_8])^{(i+j \bmod 2)}$. Then,

$$\mathcal{J} = \underbrace{\mathrm{sp}\{1+\tau, (1+\tau)a\}}_{\mathcal{J}^{(0)}} \oplus \underbrace{\mathrm{sp}\{(1+\tau)b, (1+\tau)ab\}}_{\mathcal{J}^{(1)}} \tag{25}$$

and $\mathcal{J}^{(i)}\mathcal{J}^{(j)} \subseteq \mathcal{J}^{(i+j \bmod 2)}$. The quotient algebra decomposes as

$$\mathbb{R}[Q_8]/\mathcal{J} = \underbrace{\mathrm{sp}\{1+\mathcal{J}, a+\mathcal{J}\}}_{(\mathbb{R}[Q_8]/\mathcal{J})^{(0)}} \oplus \underbrace{\mathrm{sp}\{b+\mathcal{J}, ab+\mathcal{J}\}}_{(\mathbb{R}[Q_8]/\mathcal{J})^{(1)}}, \tag{26}$$

and it has the following multiplication table:

Table 3. Multiplication table for $\mathbb{R}[Q_8]/\mathcal{J}$ based on H_1

	$1+\mathcal{J}$	$a+\mathcal{J}$	$b+\mathcal{J}$	$ab+\mathcal{J}$
$1+\mathcal{J}$	$1+\mathcal{J}$	$a+\mathcal{J}$	$b+\mathcal{J}$	$ab+\mathcal{J}$
$a+\mathcal{J}$	$a+\mathcal{J}$	$-1+\mathcal{J}$	$ab+\mathcal{J}$	$-b+\mathcal{J}$
$b+\mathcal{J}$	$b+\mathcal{J}$	$-ab+\mathcal{J}$	$-1+\mathcal{J}$	$a+\mathcal{J}$
$ab+\mathcal{J}$	$ab+\mathcal{J}$	$b+\mathcal{J}$	$-a+\mathcal{J}$	$-1+\mathcal{J}$

Thus, $(\mathbb{R}[Q_8]/\mathcal{J})^{(i)}(\mathbb{R}[Q_8]/\mathcal{J})^{(j)} \subseteq (\mathbb{R}[Q_8]/\mathcal{J})^{(i+j \bmod 2)}$, $i, j = 0, 1$. Define an $\mathbb{R}$-algebra map $\varphi : \mathbb{R}[Q_8]/\mathcal{J} \to C\ell_{0,2}$ as

$$1+\mathcal{J} \mapsto 1, \quad a+\mathcal{J} \mapsto \mathbf{e}_1\mathbf{e}_2, \quad b+\mathcal{J} \mapsto \mathbf{e}_1, \quad ab+\mathcal{J} \mapsto \mathbf{e}_2. \tag{27}$$

So, φ is a graded algebra isomorphism since $\varphi\left((\mathbb{R}[Q_8]/\mathcal{J})^{(i)}\right) \subseteq (C\ell_{0,2})^{(i)}$. Notice that in order to properly define φ, we must have equality of orders of the group elements: $|a| = |\mathbf{e}_1\mathbf{e}_2| = 4$, $|b| = |\mathbf{e}_1| = 4$, and $|ab| = |\mathbf{e}_2| = 4$ in Q_8 and $G_{0,2}$, respectively.

5.5.2. Case $H = H_2$: Since $Q_8 = H_2 \dot\cup H_2 a = \{1, \tau, b, \tau b\} \dot\cup \{a, \tau a, ab, \tau ab\}$,

$$\mathbb{R}[Q_8] = \underbrace{\mathrm{sp}\{1, \tau, b, \tau b\}}_{(\mathbb{R}[Q_8])^{(0)}} \oplus \underbrace{\mathrm{sp}\{a, \tau a, ab, \tau ab\}}_{(\mathbb{R}[Q_8])^{(1)}} \tag{28}$$

and $(\mathbb{R}[Q_8])^{(i)}(\mathbb{R}[Q_8])^{(j)} \subseteq (\mathbb{R}[Q_8])^{(i+j \text{ mod } 2)}$. Then,

$$\mathcal{J} = \underbrace{\mathrm{sp}\{1+\tau, (1+\tau)b\}}_{\mathcal{J}^{(0)}} \oplus \underbrace{\mathrm{sp}\{(1+\tau)a, (1+\tau)ab\}}_{\mathcal{J}^{(1)}} \tag{29}$$

and $\mathcal{J}^{(i)}\mathcal{J}^{(j)} \subseteq \mathcal{J}^{(i+j \text{ mod } 2)}$. The quotient algebra decomposes as

$$\mathbb{R}[Q_8]/\mathcal{J} = \underbrace{\mathrm{sp}\{1+\mathcal{J}, b+\mathcal{J}\}}_{(\mathbb{R}[Q_8]/\mathcal{J})^{(0)}} \oplus \underbrace{\mathrm{sp}\{a+\mathcal{J}, ab+\mathcal{J}\}}_{(\mathbb{R}[Q_8]/\mathcal{J})^{(1)}}, \tag{30}$$

and it has the following multiplication table:

TABLE 4. Multiplication table for $\mathbb{R}[Q_8]/\mathcal{J}$ based on H_2

	$1+\mathcal{J}$	$b+\mathcal{J}$	$a+\mathcal{J}$	$ab+\mathcal{J}$
$1+\mathcal{J}$	$1+\mathcal{J}$	$b+\mathcal{J}$	$a+\mathcal{J}$	$ab+\mathcal{J}$
$b+\mathcal{J}$	$b+\mathcal{J}$	$-1+\mathcal{J}$	$-ab+\mathcal{J}$	$a+\mathcal{J}$
$a+\mathcal{J}$	$a+\mathcal{J}$	$ab+\mathcal{J}$	$-1+\mathcal{J}$	$-b+\mathcal{J}$
$ab+\mathcal{J}$	$ab+\mathcal{J}$	$-a+\mathcal{J}$	$b+\mathcal{J}$	$-1+\mathcal{J}$

Thus, $(\mathbb{R}[Q_8]/\mathcal{J})^{(i)}\,(\mathbb{R}[Q_8]/\mathcal{J})^{(j)} \subseteq (\mathbb{R}[Q_8]/\mathcal{J})^{(i+j \text{ mod } 2)}$, $i, j = 0, 1$. Define an $\mathbb{R}$-algebra map $\varphi : \mathbb{R}[Q_8]/\mathcal{J} \to C\ell_{0,2}$ as

$$1+\mathcal{J} \mapsto 1, \quad b+\mathcal{J} \mapsto \mathbf{e}_1\mathbf{e}_2, \quad a+\mathcal{J} \mapsto \mathbf{e}_2, \quad ab+\mathcal{J} \mapsto \mathbf{e}_1. \tag{31}$$

So, again φ is a graded isomorphism and $|b| = |\mathbf{e}_1\mathbf{e}_2| = 4$, $|a| = |\mathbf{e}_2| = 4$, and $|ab| = |\mathbf{e}_1| = 4$ in Q_8 and $G_{0,2}$, respectively.

5.5.3. Case $H = H_3$: Since $Q_8 = H_3 \dot\cup H_3 a = \{1, \tau, ab, \tau ab\} \dot\cup \{a, \tau a, b, \tau b\}$,

$$\mathbb{R}[Q_8] = \underbrace{\mathrm{sp}\{1, \tau, ab, \tau ab\}}_{(\mathbb{R}[Q_8])^{(0)}} \oplus \underbrace{\mathrm{sp}\{a, \tau a, b, \tau b\}}_{(\mathbb{R}[Q_8])^{(1)}} \tag{32}$$

and $(\mathbb{R}[Q_8])^{(i)}(\mathbb{R}[Q_8])^{(j)} \subseteq (\mathbb{R}[Q_8])^{(i+j \text{ mod } 2)}$. Then,

$$\mathcal{J} = \underbrace{\mathrm{sp}\{1+\tau, (1+\tau)ab\}}_{\mathcal{J}^{(0)}} \oplus \underbrace{\mathrm{sp}\{(1+\tau)a, (1+\tau)b\}}_{\mathcal{J}^{(1)}} \tag{33}$$

and $\mathcal{J}^{(i)}\mathcal{J}^{(j)} \subseteq \mathcal{J}^{(i+j \text{ mod } 2)}$. The quotient algebra decomposes as

$$\mathbb{R}[Q_8]/\mathcal{J} = \underbrace{\mathrm{sp}\{1+\mathcal{J}, b+\mathcal{J}\}}_{(\mathbb{R}[Q_8]/\mathcal{J})^{(0)}} \oplus \underbrace{\mathrm{sp}\{a+\mathcal{J}, ab+\mathcal{J}\}}_{(\mathbb{R}[Q_8]/\mathcal{J})^{(1)}}, \tag{34}$$

and it has the following multiplication table:

TABLE 5. Multiplication table for $\mathbb{R}[Q_8]/\mathcal{J}$ based on H_3

	$1+\mathcal{J}$	$ab+\mathcal{J}$	$a+\mathcal{J}$	$b+\mathcal{J}$
$1+\mathcal{J}$	$1+\mathcal{J}$	$ab+\mathcal{J}$	$a+\mathcal{J}$	$b+\mathcal{J}$
$ab+\mathcal{J}$	$ab+\mathcal{J}$	$-1+\mathcal{J}$	$b+\mathcal{J}$	$-a+\mathcal{J}$
$a+\mathcal{J}$	$a+\mathcal{J}$	$-b+\mathcal{J}$	$-1+\mathcal{J}$	$ab+\mathcal{J}$
$ab+\mathcal{J}$	$b+\mathcal{J}$	$a+\mathcal{J}$	$-ab+\mathcal{J}$	$-1+\mathcal{J}$

Thus, $(\mathbb{R}[Q_8]/\mathcal{J})^{(i)}\,(\mathbb{R}[Q_8]/\mathcal{J})^{(j)} \subseteq (\mathbb{R}[Q_8]/\mathcal{J})^{(i+j \bmod 2)}$, $i,j=0,1$. Define an $\mathbb{R}$-algebra map $\varphi : \mathbb{R}[Q_8]/\mathcal{J} \to C\ell_{0,2}$ as

$$1+\mathcal{J} \mapsto 1, \quad ab+\mathcal{J} \mapsto \mathbf{e}_1\mathbf{e}_2, \quad a+\mathcal{J} \mapsto \mathbf{e}_1, \quad b+\mathcal{J} \mapsto \mathbf{e}_2. \tag{35}$$

Again, φ is graded and $|ab| = |\mathbf{e}_1\mathbf{e}_2| = 4$, $|a| = |\mathbf{e}_1| = 4$, and $|b| = |\mathbf{e}_2| = 4$ in Q_8 and $G_{0,2}$, respectively.

In conclusion, we have proven the following.

Theorem 5. *There is a graded algebra isomorphism $\mathbb{R}[Q_8]/\mathcal{J} \cong C\ell_{0,2}$. The isomorphism does not depend on the choice of a maximal subgroup H in Q_8 used to define the graded algebra structure in $\mathbb{R}[Q_8]/\mathcal{J}$.*

5.6. Are the isomorphisms $\mathbb{R}[D_8]/\mathcal{J} \cong C\ell_{1,1}$ and $\mathbb{R}[D_8]/\mathcal{J} \cong C\ell_{2,0}$ graded?

Recall presentation (2a) of D_8 and a multiplication Table 19 of D_8 in Appendix D. D_8 has three maximal subgroups:

$$H_1 = \langle \tau, b\rangle = \{1,\tau,b,\tau b\} \cong D_4, \quad H_2 = \langle \tau, ab\rangle = \{1,\tau,ab,\tau ab\} \cong D_4,$$
$$H_3 = \langle a\rangle = \{1,\tau,a,\tau a\} \cong C_4.$$

Each subgroup is of index 2, hence it is normal, and their intersection is $\Phi(D_8) = D_8' = Z(D_8) = \{1,\tau\}$ since D_8 is extra-special. Let H be any of these three subgroups. In the following, we will base the graded structure of $\mathbb{R}[D_8]$, $\mathcal{J}$ and $\mathbb{R}[D_8]/\mathcal{J}$ on H, and consider whether the isomorphisms $\mathbb{R}[D_8]/\mathcal{J} \cong C\ell_{2,0}$ and $\mathbb{R}[D_8]/\mathcal{J} \cong C\ell_{1,1}$ are graded.

5.6.1. Case $H = H_1$: Since $D_8 = H_1 \dot\cup H_1 a = \{1,\tau,b,\tau b\} \dot\cup \{a,\tau a,ab,\tau ab\}$,

$$\mathbb{R}[D_8] = \underbrace{\mathrm{sp}\{1,\tau,b,\tau b\}}_{(\mathbb{R}[D_8])^{(0)}} \oplus \underbrace{\mathrm{sp}\{a,\tau a,ab,\tau ab\}}_{(\mathbb{R}[D_8])^{(1)}} \tag{36}$$

and $(\mathbb{R}[D_8])^{(i)}(\mathbb{R}[D_8])^{(j)} \subseteq (\mathbb{R}[D_8])^{(i+j \bmod 2)}$. Then,

$$\mathcal{J} = \underbrace{\mathrm{sp}\{1+\tau,(1+\tau)b\}}_{\mathcal{J}^{(0)}} \oplus \underbrace{\mathrm{sp}\{(1+\tau)a,(1+\tau)ab\}}_{\mathcal{J}^{(1)}} \tag{37}$$

and $\mathcal{J}^{(i)}\mathcal{J}^{(j)} \subseteq \mathcal{J}^{(i+j \bmod 2)}$. The quotient algebra decomposes as

$$\mathbb{R}[D_8]/\mathcal{J} = \underbrace{\mathrm{sp}\{1+\mathcal{J},b+\mathcal{J}\}}_{(\mathbb{R}[D_8]/\mathcal{J})^{(0)}} \oplus \underbrace{\mathrm{sp}\{a+\mathcal{J},ab+\mathcal{J}\}}_{(\mathbb{R}[D_8]/\mathcal{J})^{(1)}}, \tag{38}$$

and it has the following multiplication table:

Table 6. Multiplication table for $\mathbb{R}[D_8]/\mathcal{J}$ based on H_1

	$1+\mathcal{J}$	$b+\mathcal{J}$	$a+\mathcal{J}$	$ab+\mathcal{J}$
$1+\mathcal{J}$	$1+\mathcal{J}$	$b+\mathcal{J}$	$a+\mathcal{J}$	$ab+\mathcal{J}$
$b+\mathcal{J}$	$b+\mathcal{J}$	$1+\mathcal{J}$	$-ab+\mathcal{J}$	$-a+\mathcal{J}$
$a+\mathcal{J}$	$a+\mathcal{J}$	$ab+\mathcal{J}$	$-1+\mathcal{J}$	$-b+\mathcal{J}$
$ab+\mathcal{J}$	$ab+\mathcal{J}$	$a+\mathcal{J}$	$b+\mathcal{J}$	$1+\mathcal{J}$

Thus, $(\mathbb{R}[D_8]/\mathcal{J})^{(i)}\,(\mathbb{R}[D_8]/\mathcal{J})^{(j)} \subseteq (\mathbb{R}[D_8]/\mathcal{J})^{(i+j \bmod 2)}$, $i,j=0,1$. Define an $\mathbb{R}$-algebra map $\varphi : \mathbb{R}[D_8]/\mathcal{J} \to C\ell_{2,0}$ as

$$1+\mathcal{J} \mapsto 1, \quad b+\mathcal{J} \mapsto \mathbf{e}_1, \quad a+\mathcal{J} \mapsto \mathbf{e}_1\mathbf{e}_2, \quad ab+\mathcal{J} \mapsto -\mathbf{e}_2. \tag{39}$$

So, while φ is an $\mathbb{R}$-isomorphism, it is not graded since the even element $b+\mathcal{J}$ is mapped into the odd element $\mathbf{e}_1$. Notice that in order to properly define φ as the $\mathbb{R}$-isomorphism, we must have equality of orders: $|b| = |\mathbf{e}_1| = 2$, $|a| = |\mathbf{e}_1\mathbf{e}_2| = 4$, and $|ab| = |-\mathbf{e}_2| = 2$ in D_8 and $G_{2,0}$, respectively. Thus, $a+\mathcal{J}$ cannot be mapped to $\mathbf{e}_1$ or $\mathbf{e}_2$ for that reason. The only flexibility in defining φ is to switch $\mathbf{e}_1$ and $\mathbf{e}_2$ in (39) (modulo -1). Thus, the isomorphism $\mathbb{R}[D_8]/\mathcal{J} \cong C\ell_{2,0}$ is not graded when $H = H_1$.

However, the isomorphism $\mathbb{R}[D_8]/\mathcal{J} \cong C\ell_{1,1}$ is graded. Define an $\mathbb{R}$-algebra map $\varphi : \mathbb{R}[D_8]/\mathcal{J} \to C\ell_{1,1}$ as

$$1+\mathcal{J} \mapsto 1, \quad b+\mathcal{J} \mapsto \mathbf{e}_1\mathbf{e}_2, \quad a+\mathcal{J} \mapsto \mathbf{e}_2, \quad ab+\mathcal{J} \mapsto \mathbf{e}_1. \tag{40}$$

So, φ is graded since $\varphi\left((\mathbb{R}[D_8]/\mathcal{J})^{(i)}\right) \subseteq (C\ell_{1,1})^{(i)}$ and $|b| = |\mathbf{e}_1\mathbf{e}_2| = 2$, $|a| = |\mathbf{e}_2| = 4$, and $|ab| = |\mathbf{e}_1| = 2$ in D_8 and $G_{1,1}$, respectively.

5.6.2. Case $H = H_2$: Since $D_8 = H_2 \,\dot\cup\, H_2 a = \{1,\tau,ab,\tau ab\} \,\dot\cup\, \{a,\tau a,b,\tau b\}$,

$$\mathbb{R}[D_8] = \underbrace{\operatorname{sp}\{1,\tau,ab,\tau ab\}}_{(\mathbb{R}[D_8])^{(0)}} \oplus \underbrace{\operatorname{sp}\{a,\tau a,b,\tau b\}}_{(\mathbb{R}[D_8])^{(1)}} \tag{41}$$

and $(\mathbb{R}[D_8])^{(i)}(\mathbb{R}[D_8])^{(j)} \subseteq (\mathbb{R}[D_8])^{(i+j \bmod 2)}$. Then,

$$\mathcal{J} = \underbrace{\operatorname{sp}\{1+\tau,(1+\tau)ab\}}_{\mathcal{J}^{(0)}} \oplus \underbrace{\operatorname{sp}\{(1+\tau)a,(1+\tau)b\}}_{\mathcal{J}^{(1)}} \tag{42}$$

and $\mathcal{J}^{(i)}\mathcal{J}^{(j)} \subseteq \mathcal{J}^{(i+j \bmod 2)}$. The quotient algebra decomposes as

$$\mathbb{R}[D_8]/\mathcal{J} = \underbrace{\operatorname{sp}\{1+\mathcal{J},ab+\mathcal{J}\}}_{(\mathbb{R}[D_8]/\mathcal{J})^{(0)}} \oplus \underbrace{\operatorname{sp}\{a+\mathcal{J},b+\mathcal{J}\}}_{(\mathbb{R}[D_8]/\mathcal{J})^{(1)}}, \tag{43}$$

and it has the following multiplication table:

TABLE 7. Multiplication table for $\mathbb{R}[D_8]/\mathcal{J}$ based on H_2

	$1+\mathcal{J}$	$ab+\mathcal{J}$	$a+\mathcal{J}$	$b+\mathcal{J}$
$1+\mathcal{J}$	$1+\mathcal{J}$	$ab+\mathcal{J}$	$a+\mathcal{J}$	$b+\mathcal{J}$
$ab+\mathcal{J}$	$ab+\mathcal{J}$	$1+\mathcal{J}$	$b+\mathcal{J}$	$a+\mathcal{J}$
$a+\mathcal{J}$	$a+\mathcal{J}$	$-b+\mathcal{J}$	$-1+\mathcal{J}$	$ab+\mathcal{J}$
$b+\mathcal{J}$	$b+\mathcal{J}$	$-a+\mathcal{J}$	$-ab+\mathcal{J}$	$1+\mathcal{J}$

Thus, $(\mathbb{R}[D_8]/\mathcal{J})^{(i)}\,(\mathbb{R}[D_8]/\mathcal{J})^{(j)} \subseteq (\mathbb{R}[D_8]/\mathcal{J})^{(i+j \bmod 2)}$, $i,j=0,1$. Define an $\mathbb{R}$-algebra map $\varphi : \mathbb{R}[D_8]/\mathcal{J} \to C\ell_{2,0}$ as

$$1+\mathcal{J} \mapsto 1, \quad ab+\mathcal{J} \mapsto \mathbf{e}_1, \quad a+\mathcal{J} \mapsto \mathbf{e}_1\mathbf{e}_2, \quad b+\mathcal{J} \mapsto \mathbf{e}_2. \tag{44}$$

So, while φ is an $\mathbb{R}$-isomorphism of algebras, it is not graded since the even element $ab+\mathcal{J}$ is mapped into the odd element $\mathbf{e}_1$. The orders of the group elements must match: $|ab| = |\mathbf{e}_1| = 2$, $|a| = |\mathbf{e}_1\mathbf{e}_2| = 4$, and $|b| = |\mathbf{e}_2| = 2$ in D_8 and $G_{2,0}$, respectively. Thus, $a+\mathcal{J}$ cannot be mapped to $\mathbf{e}_1$ or $\mathbf{e}_2$ for that reason. Thus, the isomorphism $\mathbb{R}[D_8]/\mathcal{J} \cong C\ell_{2,0}$ is not graded when $H = H_2$.

However, the isomorphism $\mathbb{R}[D_8]/\mathcal{J} \cong C\ell_{1,1}$ is graded. Define an $\mathbb{R}$-algebra map $\varphi : \mathbb{R}[D_8]/\mathcal{J} \to C\ell_{1,1}$ as

$$1+\mathcal{J} \mapsto 1, \quad ab+\mathcal{J} \mapsto \mathbf{e}_1\mathbf{e}_2, \quad a+\mathcal{J} \mapsto \mathbf{e}_2, \quad b+\mathcal{J} \mapsto -\mathbf{e}_1. \tag{45}$$

So, φ is a graded algebra isomorphism and $|ab| = |\mathbf{e}_1\mathbf{e}_2| = 2$, $|a| = |\mathbf{e}_2| = 4$, and $|b| = |-\mathbf{e}_1| = 2$ in D_8 and $G_{1,1}$, respectively.

5.6.3. Case $H = H_3$: Since $D_8 = H_3 \,\dot\cup\, H_3 b = \{1,\tau,a,\tau a\} \,\dot\cup\, \{b, ab, \tau b, \tau ab\}$,

$$\mathbb{R}[D_8] = \underbrace{\mathrm{sp}\{1,\tau,a,\tau a\}}_{(\mathbb{R}[D_8])^{(0)}} \oplus \underbrace{\mathrm{sp}\{b,ab,\tau b,\tau ab\}}_{(\mathbb{R}[D_8])^{(1)}} \tag{46}$$

and $(\mathbb{R}[D_8])^{(i)}(\mathbb{R}[D_8])^{(j)} \subseteq (\mathbb{R}[D_8])^{(i+j \bmod 2)}$. Then,

$$\mathcal{J} = \underbrace{\mathrm{sp}\{1+\tau,(1+\tau)a\}}_{\mathcal{J}^{(0)}} \oplus \underbrace{\mathrm{sp}\{(1+\tau)b,(1+\tau)ab\}}_{\mathcal{J}^{(1)}} \tag{47}$$

and $\mathcal{J}^{(i)}\mathcal{J}^{(j)} \subseteq \mathcal{J}^{(i+j \bmod 2)}$. The quotient algebra decomposes as

$$\mathbb{R}[D_8]/\mathcal{J} = \underbrace{\mathrm{sp}\{1+\mathcal{J},a+\mathcal{J}\}}_{(\mathbb{R}[D_8]/\mathcal{J})^{(0)}} \oplus \underbrace{\mathrm{sp}\{b+\mathcal{J},ab+\mathcal{J}\}}_{(\mathbb{R}[D_8]/\mathcal{J})^{(1)}}, \tag{48}$$

and it has the following multiplication table:

TABLE 8. Multiplication table for $\mathbb{R}[D_8]/\mathcal{J}$ based on H_3

	$1+\mathcal{J}$	$a+\mathcal{J}$	$b+\mathcal{J}$	$ab+\mathcal{J}$
$1+\mathcal{J}$	$1+\mathcal{J}$	$a+\mathcal{J}$	$b+\mathcal{J}$	$ab+\mathcal{J}$
$a+\mathcal{J}$	$a+\mathcal{J}$	$-1+\mathcal{J}$	$ab+\mathcal{J}$	$-b+\mathcal{J}$
$b+\mathcal{J}$	$b+\mathcal{J}$	$-ab+\mathcal{J}$	$1+\mathcal{J}$	$-a+\mathcal{J}$
$ab+\mathcal{J}$	$ab+\mathcal{J}$	$b+\mathcal{J}$	$a+\mathcal{J}$	$1+\mathcal{J}$

Thus, $(\mathbb{R}[D_8]/\mathcal{J})^{(i)}\,(\mathbb{R}[D_8]/\mathcal{J})^{(j)} \subseteq (\mathbb{R}[D_8]/\mathcal{J})^{(i+j \bmod 2)}$, $i,j = 0,1$. Define an $\mathbb{R}$-algebra map $\varphi : \mathbb{R}[D_8]/\mathcal{J} \to C\ell_{2,0}$ as

$$1+\mathcal{J} \mapsto 1, \quad a+\mathcal{J} \mapsto \mathbf{e}_1\mathbf{e}_2, \quad b+\mathcal{J} \mapsto \mathbf{e}_1, \quad ab+\mathcal{J} \mapsto -\mathbf{e}_2. \tag{49}$$

So, φ is graded and $|a| = |\mathbf{e}_1\mathbf{e}_2| = 4$, $|b| = |\mathbf{e}_1| = 2$, and $|ab| = |-\mathbf{e}_2| = 2$ in D_8 and $G_{2,0}$, respectively.

However, the isomorphism $\mathbb{R}[D_8]/\mathcal{J} \cong C\ell_{1,1}$ is not graded. Define an $\mathbb{R}$-algebra map $\varphi : \mathbb{R}[D_8]/\mathcal{J} \to C\ell_{1,1}$ as

$$1+\mathcal{J} \mapsto 1, \quad a+\mathcal{J} \mapsto \mathbf{e}_2, \quad b+\mathcal{J} \mapsto \mathbf{e}_1, \quad ab+\mathcal{J} \mapsto -\mathbf{e}_1\mathbf{e}_2. \tag{50}$$

So, φ must preserve orders of group elements namely $|a| = |\mathbf{e}_2| = 4$, $|b| = |\mathbf{e}_1| = 2$, and $|ab| = |-\mathbf{e}_1\mathbf{e}_2| = 2$ in D_8 and $G_{1,1}$, respectively. Thus, it must map an even element a into the odd element $\mathbf{e}_2$ (or $-\mathbf{e}_2$).

In conclusion, we have proven the following.

Theorem 6. *For each of the three maximal subgroups H in D_8 used to define the graded algebra structure in $\mathbb{R}[D_8]\mathcal{J}$, exactly one of the two $\mathbb{R}$-algebra isomorphisms $\mathbb{R}[D_8]/\mathcal{J} \cong C\ell_{2,0}$ or $\mathbb{R}[D_8]/\mathcal{J} \cong C\ell_{1,1}$ is graded.*

We conclude this section with a conjecture.

Conjecture. *Let C be any of the five Salingaros isomorphism classes.*

(i) *If there is a single group $G_{p,q}$ in C, then the isomorphism $\mathbb{R}[G_{p,q}]/\mathcal{J} \cong C\ell_{p,q}$ is $\mathbb{Z}_2$-graded.*

(ii) *When there is more than one group $G_{p,q}$ in C, there exists exactly one group $G_{p,q}$ in C such that the isomorphism $\mathbb{R}[G_{p,q}/\mathcal{J} \cong C\ell_{p,q}$ is $\mathbb{Z}_2$-graded while for all other groups in C, the isomorphism is not graded.*

6. Conclusions and questions

We can summarize our results as follows:

(a) Every Clifford algebra $C\ell_{p,q}$ is $\mathbb{R}$-isomorphic to one of the quotient algebras:

  (i) $\mathbb{R}[N_{\text{even}}]/\mathcal{J}$, $\mathbb{R}[N_{\text{odd}}]/\mathcal{J}$, $\mathbb{R}[S_k]/\mathcal{J}$ when $C\ell_{p,q}$ is simple, and

  (ii) $\mathbb{R}[\Omega_{\text{even}}]/\mathcal{J}$, $\mathbb{R}[\Omega_{\text{odd}}]/\mathcal{J}$ when $C\ell_{p,q}$ is semisimple,

of Salingaros vee groups N_{even}, N_{odd}, S_k, Ω_{even}, and Ω_{odd} modulo the ideal $\mathcal{J} = (1+\tau)$ for a central element τ of order 2.

(b) The group algebra $\mathbb{R}[G_{p,q}]$ is $\mathbb{Z}_2$-graded and the ideal $\mathcal{J} = (1+\tau)$, where τ is a central involution, is homogeneous and $\mathbb{Z}_2$-graded.
(c) In some cases, there is a graded algebra isomorphism between $\mathbb{R}[G_{p,q}]/\mathcal{J}$ and $C\ell_{p,q}$. We conjecture that each Clifford algebra $C\ell_{p,q}$ is graded-isomorphic with $\mathbb{R}[G_{p,q}]/\mathcal{J}$.
(d) We have presented two algorithms that allow one to factor each Salingaros vee group into a central product of its subgroups.

The following questions remain:

(a) How does the group structure of $G_{p,q}$, e.g., presence of normal subgroups and the central product structure, carry over to the algebra structure of $C\ell_{p,q}$? If so, how?
(b) Apply the character theory and representation theory methods of 2-groups to the group algebras $\mathbb{R}[G_{p,q}]$ and their quotients $\mathbb{R}[G_{p,q}]/\mathcal{J}$, and hence to the Clifford algebras $C\ell_{p,q}$. In particular, relate, if possible, primitive idempotents in $C\ell_{p,q}$ to the irreducible characters of $G_{p,q}$.

Appendix A. Two special cases: $G_{0,1} \cong S_0$ and $G_{1,0} \cong \Omega_0$

In the Main Theorem in Section 5 and later in Lemma 5, we excluded two Clifford algebras when $p+q=1$: $C\ell_{0,1}$ and $C\ell_{1,0}$. However,

$$G_{0,1} \cong S_0 \cong \langle g_1, \tau \mid (g_1)^2 = \tau, \tau^2 = 1, g_1\tau = \tau g_1 \rangle \cong C_4, \tag{51}$$

$$G_{1,0} \cong \Omega_0 \cong \langle g_1, \tau \mid (g_1)^2 = 1, \tau^2 = 1, g_1\tau = \tau g_1 \rangle \cong C_2 \times C_2, \tag{52}$$

where, in both cases, the group elements are $1, g_1, \tau, g_1\tau$. Let G denote either group and let $H = \langle \tau \rangle = \{1, \tau\} < G$. It is easy to show that both group algebras are graded with the following two homogeneous parts:

$$\mathbb{R}[G] = (\mathbb{R}[G])^{(0)} \oplus (\mathbb{R}[G])^{(1)} = \mathrm{sp}\{1, \tau\} \oplus \mathrm{sp}\{g_1, g_1\tau\} \tag{53}$$

Define a homogeneous ideal $\mathcal{J} = (1+\tau)$ in $\mathbb{R}[G]$. Then, in both cases,

$$\mathcal{J} = \mathcal{J}^{(0)} \oplus \mathcal{J}^{(1)} = \mathrm{sp}\{1+\tau\} \oplus \mathrm{sp}\{g_1(1+\tau)\}. \tag{54}$$

Furthermore, the quotient algebra $\mathbb{R}[G]/\mathcal{J}$ is also graded:

$$\mathbb{R}[G]/\mathcal{J} = (\mathbb{R}[G]/\mathcal{J})^{(0)} \oplus (\mathbb{R}[G]/\mathcal{J})^{(1)} = \mathrm{sp}\{1+\mathcal{J}\} \oplus \mathrm{sp}\{g_1+\mathcal{J}\} \tag{55}$$

with

$$(g_1+\mathcal{J})^2 = (g_1)^2 + \mathcal{J} = \begin{cases} -1+\mathcal{J}, & \text{when } G = G_{0,1}; \\ 1+\mathcal{J}, & \text{when } G = G_{1,0}. \end{cases} \tag{56}$$

We can now define two $\mathbb{R}$-algebra isomorphisms

$$\varphi_1 : \mathbb{R}[G_{0,1}]/\mathcal{J} \to C\ell_{0,1} \quad \text{and} \quad \varphi_2 : \mathbb{R}[G_{1,0}]/\mathcal{J} \to C\ell_{1,0} \quad \text{as}$$
$$1+\mathcal{J} \mapsto 1, \quad g_1+\mathcal{J} \mapsto \mathbf{e}_1, \quad \tau+\mathcal{J} \mapsto -1 \tag{57}$$

which happen to be graded algebra isomorphisms.

Appendix B. Groups $G_{p,q}$ as central products

In this appendix we write Salingaros vee groups $G_{p,q}$ as internal central products for $3 \le p+q \le 9$. In particular, we show generators for all subgroups appearing in the central products.

We present two algorithms by which we can factor any vee group into a central product of subgroups. Recall that the central product is commutative and associative, and that the subgroups appearing in the factorization are unique up to an isomorphism. Our standard presentations for D_8 and Q_8 is as follows:

$$N_1 \cong D_8 = \langle a, b\rangle = \langle a, b \mid a^4 = b^2 = 1, b^{-1}ab = a^{-1}\rangle, \tag{58}$$

$$N_2 \cong Q_8 = \langle a, b\rangle = \langle a, b \mid a^4 = 1, a^2 = b^2, b^{-1}ab = a^{-1}\rangle. \tag{59}$$

Recall also that the centralizer $C_G(H)$ of a subgroup of G is defined as

$$C_G(H) = \{x \in G \mid xh = hx \text{ for all } h \in H\}. \tag{60}$$

Note that $C_G(H)$ is a subgroup of G, $C_G(G) = Z(G)$, and $C_G(H) \le N_G(H)$, the normalizer of H in G [16].

Algorithm 1: Algorithm to factor N_{2k-1} as central product, $k \ge 2$.

1. **Data:** $G = G_{p,q}, p+q \ge 3$
2. **Result:** Factorization $G \cong N_{2k-1} \cong N_1^{\circ k} = D_8^{(k)} \circ \cdots \circ D_8^{(2)} \circ D_8^{(1)}$.
3. Find order structure of G:
   L_2 = list of elements in G of order 2,
   L_4 = list of elements in G of order 4.
4. Choose generators a and b for $D_8^{(1)}$ from L_4 and L_2.
5. Generate $D_8^{(1)} = \langle a, b\rangle$.
6. Compute centralizer $K = C_G(D_8^{(1)})$.
7. **for** $i \leftarrow 2$ **to** k **do**
   - Find order structure of K;
     L_2 = list of elements in K of order 2;
     L_4 = list of elements in K of order 4;
   - Choose generators a and b for $D_8^{(i)}$ from L_4 and L_2;
   - Generate $D_8^{(i)} = \langle a, b\rangle$;
   - Compute $D_8^{(i)} \circ D_8^{(i-1)} \circ \cdots \circ D_8^{(1)}$ as $D_8^{(i)}(D_8^{(i-1)} \cdots D_8^{(1)})$;
   - Find centralizer $K = C_G(D_8^{(i)} \circ D_8^{(i-1)} \circ \cdots \circ D_8^{(1)})$;

   end

   return *generators for* $D_8^{(k)}, \ldots, D_8^{(1)}$.

Example 8 (Factoring in N_5). *Let $G = G_{0,6}$ so $|G_{0,6}| = 128$ and*

$$G = G_{0,6} \cong N_5 \cong N_1^{\circ 3} = D_8^{(3)} \circ D_8^{(2)} \circ D_8^{(1)}, \quad k = 3. \tag{61}$$

where $D_8^{(i)}, i = 1, 2, 3$, are subgroups in $G_{0,6}$, each isomorphic to D_8. We execute steps 3-6 in Algorithm 1:

3. *The order structure of G is $[1, 71, 56]$ and lists L_2 and L_4 contain all elements of G of order 2 and 4 respectively:*

$$L_2 = (\mathbf{e}_{123}, \mathbf{e}_{124}, \mathbf{e}_{125}, \dots, \mathbf{e}_{234}, \dots), \quad L_4 = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \dots). \tag{62}$$

4. *We let $a = \mathbf{e}_1, b = \mathbf{e}_{234}$ so that relations (58) are satisfied.*
5. *We generate $D_8^{(1)}$ as $\langle a, b\rangle$.*
6. *We compute the centralizer $K = C_G(D_8^{(1)})$. The order of K, as expected, is 32.*

Next, we perform the do–loop in step 7 twice for $i = 2, 3$ and find generators for $D_8^{(2)}$ and $D_8^{(3)}$.

7. *The order structure of K is $[1, 19, 12]$ and lists L_2 and L_4 contain all elements of K of order 2 and 4 respectively:*

$$L_2 = (\mathbf{e}_{125}, \mathbf{e}_{126}, \mathbf{e}_{135}, \dots), \quad L_4 = (\mathbf{e}_{23}, \mathbf{e}_{24}, \mathbf{e}_{34}, \dots). \tag{63}$$

   We let $a = \mathbf{e}_{23}, b = \mathbf{e}_{125}$ so that relations (58) are satisfied.
   We generate $D_8^{(2)} = \langle a, b\rangle$.
   We compute $D_8^{(2)} \circ D_8^{(1)}$ as the product $D_8^{(2)} D_8^{(1)}$ due to the definition of the central product.
   We compute the centralizer $K = C_G(D_8^{(2)} \circ D_8^{(1)})$. The order of K is 8, as expected.

In fact, this last centralizer K is already our third dihedral group $D_8^{(3)}$ and it can be generated, as the second iteration of this do–loop confirms, by choosing $a = \mathbf{e}_{12345}$ and $b = \mathbf{e}_{146}$. Thus, we conclude, that

$$\begin{aligned} G_{0,6} \cong N_5 \cong N_1^{\circ 3} &= D_8^{(3)} \circ D_8^{(2)} \circ D_8^{(1)} \\ &= \langle \mathbf{e}_{12345}, \mathbf{e}_{146}\rangle \circ \langle \mathbf{e}_{23}, \mathbf{e}_{125}\rangle \circ \langle \mathbf{e}_1, \mathbf{e}_{234}\rangle. \end{aligned} \tag{64}$$

It is easy to check that the three sets of generators $\{\mathbf{e}_{12345}, \mathbf{e}_{146}\}$, $\{\mathbf{e}_{23}, \mathbf{e}_{125}\}$, and $\{\mathbf{e}_1, \mathbf{e}_{234}\}$ centralize each other and generate three copies of the dihedral group D_8 in $G_{0,6}$. Notice also that the Product Formula gives

$$2^5 = |N_3| = |D_8^{(2)} \circ D_8^{(1)}| = \frac{|D_8^{(2)}||D_8^{(2)}|}{|D_8^{(2)} \cap D_8^{(1)}|}, \tag{65}$$

so, as expected, $|D_8^{(2)} \cap D_8^{(1)}| = 2$. Furthermore,

$$2^7 = |N_5| = |D_8^{(2)} \circ (D_8^{(2)} \circ D_8^{(1)})| = \frac{|D_8^{(3)}||D_8^{(2)} \circ D_8^{(1)}|}{|D_8^{(3)} \cap (D_8^{(2)} \circ D_8^{(1)})|}, \tag{66}$$

so, again as expected, $|D_8^{(2)} \cap (D_8^{(2)} \circ D_8^{(1)})| = 2$. This implies that

$$D_8^{(3)} \cap D_8^{(2)} \cap D_8^{(1)} = Z(G_{0,6}) \cong C_2. \tag{67}$$

Algorithm 2 is a small modification of Algorithm 1. The modification is in the steps 4–6.

Algorithm 2: Algorithm to factor N_{2k} as central product, $k \geq 2$.

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1. **Data:** $G = G_{p,q}, p+q \geq 3$
2. **Result:** Factorization $G \cong N_{2k} \cong N_1^{\circ(k-1)} \circ N_2 = D_8^{(k)} \circ \cdots \circ D_8^{(2)} \circ Q_8$.
3. Find order structure of G:
   $L_2 =$ list of elements in G of order 2,
   $L_4 =$ list of elements in G of order 4.
4. Choose generators a and b for Q_8 from L_4 subject to (59).
5. Generate $Q_8 = \langle a, b\rangle$.
6. Compute centralizer $K = C_G(Q_8)$.
7. **for** $i \leftarrow 2$ **to** k **do**
   Find order structure of K;
   $L_2 =$ list of elements in K of order 2;
   $L_4 =$ list of elements in K of order 4;
   Choose generators a and b for $D_8^{(i)}$ from L_4 and L_2;
   Generate $D_8^{(i)} = \langle a, b\rangle$;
   Compute $D_8^{(i)} \circ D_8^{(i-1)} \circ \cdots \circ D_8^{(2)} \circ Q_8$ as $D_8^{(i)}(D_8^{(i-1)} \cdots D_8^{(2)} Q_8)$;
   Find centralizer $K = C_G(D_8^{(i)} \circ D_8^{(i-1)} \circ \cdots \circ D_8^{(2)} \circ Q_8$;
   end
   return *generators for* $D_8^{(k)}, \ldots, D_8^{(2)}, Q_8$.

Example 9 (Factoring in N_8). *We apply Algorithm 2 to $G = G_{2,6}$ of order 512. So, we are seeking the following factorization:*

$$G = G_{2,6} \cong N_8 \cong N_1^{\circ 4} \circ Q_8 = D_8^{(4)} \circ D_8^{(3)} \circ D_8^{(2)} \circ Q_8, \quad k = 4. \tag{68}$$

where $D_8^{(i)}, i = 2,3,4$, are subgroups in $G_{2,6}$, each isomorphic to D_8. We execute steps 3-6 in Algorithm 2:

3. *The order structure of G is $[1, 239, 272]$ and lists L_2 and L_4 contain all elements of G of order 2 and 4 respectively:*

$$L_2 = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_{13}, \ldots), \quad L_4 = (\mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_5, \ldots). \tag{69}$$

4. *We let $a = \mathbf{e}_3, b = \mathbf{e}_4$ so that relations (59) are satisfied.*
5. *We generate Q_8 as $\langle a, b\rangle$.*
6. *We compute the centralizer $K = C_G(Q_8)$. The order of K is 128, as expected.*

The do-loop in step 7 is performed three times for $i = 2,3,4$. In the end, we find generators for the three subgroups of $G_{2,6}$ isomorphic to D_8 that centralize each other and the subgroup isomorphic to Q_8. Thus, we conclude, that

$$\begin{aligned} G_{2,6} &\cong N_8 \cong N_1^{\circ 3} \circ N_2 \\ &= D_8^{(4)} \circ D_8^{(3)} \circ D_8^{(2)} \circ Q_8 \\ &= \langle \mathbf{e}_{34678}, \mathbf{e}_{1258}\rangle \circ \langle \mathbf{e}_{67}, \mathbf{e}_{346}\rangle \circ \langle \mathbf{e}_{12}, \mathbf{e}_{15}\rangle \circ \langle \mathbf{e}_3, \mathbf{e}_4\rangle. \end{aligned} \tag{70}$$

Obviously, $D_8^{(4)} \cap D_8^{(3)} \cap D_8^{(2)} \cap Q_8 = Z(G_{2,6})$ *as a direct consequence of the fact that* $Z(D_8) \cong Z(Q_8) \cong Z(G_{2,6}) \cong C_2$.

Algorithms 1 and 2 allow us to factor vee groups $G_{p,q}$ in the remaining three classes Ω_{2k-1}, Ω_{2k} and S_k as well. The right-most group is always the center $Z(G_{p,q})$, which is either isomorphic to $D_4 = C_2 \times C_2$ or C_4. Of course, the centralizer of each center is the whole group, thus, knowing that $D_4 \cong \langle \beta, -\beta \rangle$ while $C_4 \cong \langle \beta \rangle$, where β is the unit pseudoscalar in $C\ell_{p,q}$, one can begin by finding generators for the second-rightmost group, either $N_1 = D_8$ or $N_2 = Q_8$, as needed. Notice that since

$$|\Omega_{2k-1}| = |N_{2k-1} \circ D_4| = \frac{|N_{2k-1}||D_4|}{|N_{2k-1} \cap D_4|}, \tag{71a}$$

$$|\Omega_{2k}| = |N_{2k} \circ D_4| = \frac{|N_{2k}||D_4|}{|N_{2k} \cap D_4|}, \tag{71b}$$

$$|S_k| = |N_{2k-1} \circ C_4| = \frac{|N_{2k-1}||C_4|}{|N_{2k-1} \cap C_4|}, \tag{71c}$$

$$= |N_{2k} \circ C_4| = \frac{|N_{2k}||C_4|}{|N_{2k} \cap C_4|}, \tag{71d}$$

and $|N_{2k-1}| = |N_{2k}| = 2^{2k+1}$ and $|\Omega_{2k-1}| = |\Omega_{2k}| = |S_k| = 2^{2k+2}$, we find that the orders of all group intersections in (71) equal to 2. This implies, as expected, that the central involution τ belongs to N_{2k} or $|N_{2k-1}$, respectively. We illustrate this process in our next example.

Example 10 (Factoring in Ω_1, Ω_4, S_2).
(a) To factor a group of class Ω_{2k-1}*, for example* Ω_1*, we use Algorithm* 1*:*

$$G_{2,1} \cong \Omega_1 \cong N_1 \circ D_4 \cong D_8 \circ D_4. \tag{72}$$

The order structure of $G_{2,1}$ *is* $[1, 11, 4]$ *with elements of order* 2 *and* 4 *as*

$$L_2 = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_{13}, \ldots) \quad \textit{and} \quad L_4 = (\mathbf{e}_3, \mathbf{e}_{12}, \ldots). \tag{73}$$

Thus, we let $D_8 = \langle \mathbf{e}_3, \mathbf{e}_1 \rangle$ *and verify that* $G_{2,1} = \langle \mathbf{e}_3, \mathbf{e}_1 \rangle \circ \langle \beta, -\beta \rangle$ *where* $\beta = \mathbf{e}_{123}$ *is central of order* 2*.*

(b) To factor a group of class Ω_{2k}*, for example* Ω_4*, we use Algorithm* 2*:*

$$G_{1,4} \cong \Omega_4 \cong N_4 \circ D_4 \cong N_1 \circ N_2 \circ D_4 \cong D_8 \circ Q_8 \circ D_4. \tag{74}$$

The order structure of $G = G_{1,4}$ *is* $[1, 23, 40]$ *with elements of order* 2 *and* 4 *as*

$$L_2 = (\mathbf{e}_1, \mathbf{e}_{12}, \mathbf{e}_{13}, \ldots) \quad \textit{and} \quad L_4 = (\mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4, \ldots). \tag{75}$$

Thus, we let $Q_8 = \langle \mathbf{e}_2, \mathbf{e}_3 \rangle$*, verify that* $|Q_8 \circ D_4| = 16$*, as expected, and compute the centralizer*

$$\begin{aligned} K = C_G(Q_8 \circ D_4) = \{&\pm 1, \pm \mathbf{e}_{14}, \pm \mathbf{e}_{15}, \pm \mathbf{e}_{45}, \\ &\pm \mathbf{e}_{123}, \pm \mathbf{e}_{234}, \pm \mathbf{e}_{235}, \pm \mathbf{e}_{12345}\}. \end{aligned} \tag{76}$$

The do-loop needs to be executed once. The order structure of K *is* $[1, 11, 4]$ *with elements of order* 2 *and* 4 *as*

$$L_2 = (\mathbf{e}_{14}, \mathbf{e}_{15}, \mathbf{e}_{234}, \ldots) \quad \textit{and} \quad L_4 = (\mathbf{e}_{45}, \mathbf{e}_{123}, \ldots). \tag{77}$$

We let $D_8 = \langle \mathbf{e}_{45}, \mathbf{e}_{14} \rangle$ *and conclude that* $D_8 \circ (Q_8 \circ D_4) = G_{1,4}$ *with*

$$D_8 = \{\pm 1, \pm\mathbf{e}_{14}, \pm\mathbf{e}_{15}, \pm\mathbf{e}_{45}\}, \quad Q_8 = \{\pm 1, \pm\mathbf{e}_2, \pm\mathbf{e}_3, \pm\mathbf{e}_{23}\}, \tag{78}$$

and $D_4 = \{\pm 1, \pm\beta\}$ *where* $\beta = \mathbf{e}_{12345}$ *is central of order* 2*.*

(c) To factor a group of class S_k*, we use either algorithm depending on which factorization we seek. For example, the two ways to factor* S_2 *are:*

$$G_{0,5} \cong S_2 \cong N_3 \circ C_4 \cong N_1 \circ N_1 \circ C_4 \cong D_8^{(2)} \circ D_8^{(1)} \circ C_4, \quad \textit{or}, \tag{79a}$$

$$G_{0,5} \cong S_2 \cong N_4 \circ C_4 \cong N_1 \circ N_2 \circ C_4 \cong D_8 \circ Q_8 \circ C_4. \tag{79b}$$

Let us find factorization (79a) *with Algorithm* 1*. Recall that* $Z(S_k) = \langle \beta \rangle \cong C_4$ *since* $|\beta| = 4$ *in every group* S_k*. The order structure of* $G_{0,5}$ *is* $[1, 31, 32]$ *with elements of order* 2 *and* 4 *as*

$$L_2 = (\mathbf{e}_{123}, \mathbf{e}_{124}, \mathbf{e}_{125}, \ldots) \quad \textit{and} \quad L_4 = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \ldots). \tag{80}$$

Thus, we let $D_8^{(1)} = \langle \mathbf{e}_1, \mathbf{e}_{234} \rangle$*, verify that* $|D_8^{(1)} \circ C_4| = 16$*, and compute the centralizer*

$$K = C_G(D_8^{(1)} \circ C_4) = \{\pm 1, \pm\mathbf{e}_{23}, \pm\mathbf{e}_{24}, \pm\mathbf{e}_{34}, \\ \pm\mathbf{e}_{125}, \pm\mathbf{e}_{135}, \pm\mathbf{e}_{145}, \pm\mathbf{e}_{12345}\}. \tag{81}$$

The order structure of K *is* $[1, 7, 8]$ *with elements of order* 2 *and* 4 *as*

$$L_2 = (\mathbf{e}_{125}, \mathbf{e}_{135}, \mathbf{e}_{145}, \ldots) \quad \textit{and} \quad L_4 = (\mathbf{e}_{23}, \mathbf{e}_{24}, \mathbf{e}_{34}, \ldots). \tag{82}$$

We let $D_8^{(2)} = \langle \mathbf{e}_{23}, \mathbf{e}_{125} \rangle$ *and conclude that* $D_8^{(2)} \circ (D_8^{(1)} \circ C_4) = G_{0,5}$ *with*

$$D_8^{(2)} = \{\pm 1, \pm\mathbf{e}_{23}, \pm\mathbf{e}_{125}, \pm\mathbf{e}_{135}\}, \quad D_8^{(1)} = \{\pm 1, \pm\mathbf{e}_1, \pm\mathbf{e}_{234}, \pm\mathbf{e}_{1234}\}, \tag{83}$$

and $C_4 = \langle \beta \rangle = \{\pm 1, \pm\beta\}$ *where* $\beta = \mathbf{e}_{12345}$ *is central of order* 4*. Similarly, factorization* (79b) *can be found with Algorithm* 2*.*

In the following, we collect results of our factorization algorithms. To simplify our notation, we present copies of the dihedral group $N_1 = D_8$ and the quaternion group $N_2 = Q_8$ just by listing their generators as in $\langle a, b \rangle$. All central products are written by juxtaposition, $(N_1)^{\circ k}$ denotes the internal central product of k copies of N_1, and $D_4 = C_2 \times C_2$. Depending on the order of a pseudoscalar β, the center $Z(G_{p,q})$ of order 4 is either elementary abelian $\langle \beta, -\beta \rangle$ or cyclic $\langle \beta \rangle$. All computations have been implemented in `CLIFFORD` [1,2] and an additional package for handling groups.

TABLE 9. $p+q=3$: Groups $G_{p,q}$ of order 2^4, $\beta = \mathbf{e}_{123}$

Group	Class	Group factorization
$G_{0,3}$	$\Omega_2 \cong N_2D_4$	$\langle \mathbf{e}_1, \mathbf{e}_2 \rangle \langle \beta, -\beta \rangle$
$G_{1,2}$	$S_1 \cong N_1C_4$	$\langle \mathbf{e}_2, \mathbf{e}_1 \rangle \langle \beta \rangle$
	$S_1 \cong N_2C_4$	$\langle \mathbf{e}_2, \mathbf{e}_3 \rangle \langle \beta \rangle$
$G_{2,1}$	$\Omega_1 \cong N_1D_4$	$\langle \mathbf{e}_3, \mathbf{e}_1 \rangle \langle \beta, -\beta \rangle$
$G_{3,0}$	$S_1 \cong N_1C_4$	$\langle \mathbf{e}_{12}, \mathbf{e}_1 \rangle \langle \beta \rangle$
	$S_1 \cong N_2C_4$	$\langle \mathbf{e}_{12}, \mathbf{e}_{13} \rangle \langle \beta \rangle$

TABLE 10. $p+q=4$: Groups $G_{p,q}$ of order 2^5

Group	Class	Group factorization
$G_{0,4}$	$N_4 \cong N_1N_2$	$\langle \mathbf{e}_{34}, \mathbf{e}_{123} \rangle \langle \mathbf{e}_1, \mathbf{e}_2 \rangle$
$G_{1,3}$	$N_4 \cong N_1N_2$	$\langle \mathbf{e}_{123}, \mathbf{e}_{14} \rangle \langle \mathbf{e}_2, \mathbf{e}_3 \rangle$
$G_{2,2}$	$N_3 \cong N_1N_1$	$\langle \mathbf{e}_{134}, \mathbf{e}_{24} \rangle \langle \mathbf{e}_3, \mathbf{e}_1 \rangle$
	$N_3 \cong N_2N_2$	$\langle \mathbf{e}_{12}, \mathbf{e}_{134} \rangle \langle \mathbf{e}_3, \mathbf{e}_4 \rangle$
$G_{3,1}$	$N_3 \cong N_1N_1$	$\langle \mathbf{e}_{23}, \mathbf{e}_{1234} \rangle \langle \mathbf{e}_4, \mathbf{e}_1 \rangle$
	$N_3 \cong N_2N_2$	$\langle \mathbf{e}_{12}, \mathbf{e}_{13} \rangle \langle \mathbf{e}_4, \mathbf{e}_{123} \rangle$
$G_{4,0}$	$N_4 \cong N_1N_2$	$\langle \mathbf{e}_{123}, \mathbf{e}_4 \rangle \langle \mathbf{e}_{12}, \mathbf{e}_{13} \rangle$

TABLE 11. $p+q=5$: Groups $G_{p,q}$ of order 2^6, $\beta = \mathbf{e}_{12345}$

Group	Class	Group factorization
$G_{0,5}$	$S_2 \cong (N_1)^{\circ 2}C_4$	$\langle \mathbf{e}_{23}, \mathbf{e}_{125} \rangle \langle \mathbf{e}_1, \mathbf{e}_{234} \rangle \langle \beta \rangle$
	$S_2 \cong N_1N_2C_4$	$\langle \mathbf{e}_{34}, \mathbf{e}_{123} \rangle \langle \mathbf{e}_1, \mathbf{e}_2 \rangle \langle \beta \rangle$
$G_{1,4}$	$\Omega_4 \cong N_1N_2D_4$	$\langle \mathbf{e}_{45}, \mathbf{e}_{14} \rangle \langle \mathbf{e}_2, \mathbf{e}_3 \rangle \langle \beta, -\beta \rangle$
$G_{2,3}$	$S_2 \cong (N_1)^{\circ 2}C_4$	$\langle \mathbf{e}_{45}, \mathbf{e}_{24} \rangle \langle \mathbf{e}_3, \mathbf{e}_1 \rangle \langle \beta \rangle$
	$S_2 \cong N_1N_2C_4$	$\langle \mathbf{e}_{12}, \mathbf{e}_{15} \rangle \langle \mathbf{e}_3, \mathbf{e}_4 \rangle \langle \beta \rangle$
$G_{3,2}$	$\Omega_3 \cong (N_1)^{\circ 2}D_4$	$\langle \mathbf{e}_{23}, \mathbf{e}_{25} \rangle \langle \mathbf{e}_4, \mathbf{e}_1 \rangle \langle \beta, -\beta \rangle$
$G_{4,1}$	$S_2 \cong (N_1)^{\circ 2}C_4$	$\langle \mathbf{e}_{23}, \mathbf{e}_{125} \rangle \langle \mathbf{e}_5, \mathbf{e}_1 \rangle \langle \beta \rangle$
	$S_2 \cong N_1N_2C_4$	$\langle \mathbf{e}_{12}, \mathbf{e}_{145} \rangle \langle \mathbf{e}_5, \mathbf{e}_{123} \rangle \langle \beta \rangle$
$G_{5,0}$	$\Omega_4 \cong N_1N_2D_4$	$\langle \mathbf{e}_{45}, \mathbf{e}_4 \rangle \langle \mathbf{e}_{12}, \mathbf{e}_{13} \rangle \langle \beta, -\beta \rangle$

TABLE 12. $p+q=6$: Groups $G_{p,q}$ of order 2^7

Group	Class	Group factorization
$G_{0,6}$	$N_5 \cong (N_1)^{\circ 3}$	$\langle \mathbf{e}_{12345}, \mathbf{e}_{146} \rangle \langle \mathbf{e}_{23}, \mathbf{e}_{125} \rangle \langle \mathbf{e}_1, \mathbf{e}_{234} \rangle$
$G_{1,5}$	$N_6 \cong (N_1)^{\circ 2}N_2$	$\langle \mathbf{e}_{1456}, \mathbf{e}_{236} \rangle \langle \mathbf{e}_{45}, \mathbf{e}_{14} \rangle \langle \mathbf{e}_2, \mathbf{e}_3 \rangle$
$G_{2,4}$	$N_6 \cong (N_1)^{\circ 2}N_2$	$\langle \mathbf{e}_{12345}, \mathbf{e}_{346} \rangle \langle \mathbf{e}_{12}, \mathbf{e}_{15} \rangle \langle \mathbf{e}_3, \mathbf{e}_4 \rangle$
$G_{3,3}$	$N_5 \cong (N_1)^{\circ 3}$	$\langle \mathbf{e}_{146}, \mathbf{e}_{2356} \rangle \langle \mathbf{e}_{23}, \mathbf{e}_{25} \rangle \langle \mathbf{e}_4, \mathbf{e}_1 \rangle$
$G_{4,2}$	$N_5 \cong (N_1)^{\circ 3}$	$\langle \mathbf{e}_{2346}, \mathbf{e}_{145} \rangle \langle \mathbf{e}_{23}, \mathbf{e}_{26} \rangle \langle \mathbf{e}_5, \mathbf{e}_1 \rangle$
$G_{5,1}$	$N_6 \cong (N_1)^{\circ 2}N_2$	$\langle \mathbf{e}_{12346}, \mathbf{e}_{356} \rangle \langle \mathbf{e}_{12}, \mathbf{e}_{146} \rangle \langle \mathbf{e}_6, \mathbf{e}_{123} \rangle$
$G_{6,0}$	$N_6 \cong (N_1)^{\circ 2}N_2$	$\langle \mathbf{e}_{456}, \mathbf{e}_{1236} \rangle \langle \mathbf{e}_{45}, \mathbf{e}_4 \rangle \langle \mathbf{e}_{12}, \mathbf{e}_{13} \rangle$

TABLE 13. $p+q=7$: Groups $G_{p,q}$ of order 2^8, $\beta = \mathbf{e}_{1234567}$

Group	Class	Group factorization
$G_{0,7}$	$\Omega_5 \cong (N_1)^{\circ 3} D_4$	$\langle \mathbf{e}_{67}, \mathbf{e}_{146} \rangle \langle \mathbf{e}_{23}, \mathbf{e}_{125} \rangle \langle \mathbf{e}_1, \mathbf{e}_{234} \rangle \langle \beta, -\beta \rangle$
$G_{1,6}$	$S_3 \cong (N_1)^{\circ 3} C_4$	$\langle \mathbf{e}_{56}, \mathbf{e}_{12345} \rangle \langle \mathbf{e}_{34}, \mathbf{e}_{3567} \rangle \langle \mathbf{e}_2, \mathbf{e}_1 \rangle \langle \beta \rangle$
	$S_3 \cong (N_1)^{\circ 2} N_2 C_4$	$\langle \mathbf{e}_{67}, \mathbf{e}_{236} \rangle \langle \mathbf{e}_{45}, \mathbf{e}_{14} \rangle \langle \mathbf{e}_2, \mathbf{e}_3 \rangle \langle \beta \rangle$
$G_{2,5}$	$\Omega_6 \cong (N_1)^{\circ 2} N_2 D_4$	$\langle \mathbf{e}_{67}, \mathbf{e}_{346} \rangle \langle \mathbf{e}_{12}, \mathbf{e}_{15} \rangle \langle \mathbf{e}_3, \mathbf{e}_4 \rangle \langle \beta, -\beta \rangle$
$G_{3,4}$	$S_3 \cong (N_1)^{\circ 3} C_4$	$\langle \mathbf{e}_{67}, \mathbf{e}_{2356} \rangle \langle \mathbf{e}_{23}, \mathbf{e}_{25} \rangle \langle \mathbf{e}_4, \mathbf{e}_1 \rangle \langle \beta \rangle$
	$S_3 \cong (N_1)^{\circ 2} N_2 C_4$	$\langle \mathbf{e}_{345}, \mathbf{e}_{37} \rangle \langle \mathbf{e}_{12}, \mathbf{e}_{16} \rangle \langle \mathbf{e}_4, \mathbf{e}_5 \rangle \langle \beta \rangle$
$G_{4,3}$	$\Omega_5 \cong (N_1)^{\circ 3} D_4$	$\langle \mathbf{e}_{157}, \mathbf{e}_{47} \rangle \langle \mathbf{e}_{23}, \mathbf{e}_{26} \rangle \langle \mathbf{e}_5, \mathbf{e}_1 \rangle \langle \beta, -\beta \rangle$
$G_{5,2}$	$S_3 \cong (N_1)^{\circ 3} C_4$	$\langle \mathbf{e}_{45}, \mathbf{e}_{146} \rangle \langle \mathbf{e}_{23}, \mathbf{e}_{27} \rangle \langle \mathbf{e}_6, \mathbf{e}_1 \rangle \langle \beta \rangle$
	$S_3 \cong (N_1)^{\circ 2} N_2 C_4$	$\langle \mathbf{e}_{34}, \mathbf{e}_{12367} \rangle \langle \mathbf{e}_{12}, \mathbf{e}_{1345} \rangle \langle \mathbf{e}_6, \mathbf{e}_7 \rangle \langle \beta \rangle$
$G_{6,1}$	$\Omega_6 \cong (N_1)^{\circ 2} N_2 D_4$	$\langle \mathbf{e}_{56}, \mathbf{e}_{357} \rangle \langle \mathbf{e}_{12}, \mathbf{e}_{147} \rangle \langle \mathbf{e}_7, \mathbf{e}_{123} \rangle \langle \beta, -\beta \rangle$
$G_{7,0}$	$S_3 \cong (N_1)^{\circ 3} C_4$	$\langle \mathbf{e}_{56}, \mathbf{e}_{12345} \rangle \langle \mathbf{e}_{34}, \mathbf{e}_{3567} \rangle \langle \mathbf{e}_{12}, \mathbf{e}_1 \rangle \langle \beta \rangle$
	$S_3 \cong (N_1)^{\circ 2} N_2 C_4$	$\langle \mathbf{e}_{67}, \mathbf{e}_{1236} \rangle \langle \mathbf{e}_{45}, \mathbf{e}_4 \rangle \langle \mathbf{e}_{12}, \mathbf{e}_{13} \rangle \langle \beta \rangle$

TABLE 14. $p+q=8$: Groups $G_{p,q}$ of order 2^9

Group	Class	Group factorization
$G_{0,8}$	$N_7 \cong (N_1)^{\circ 4}$	$\langle \mathbf{e}_{14678}, \mathbf{e}_{2358} \rangle \langle \mathbf{e}_{67}, \mathbf{e}_{146} \rangle \langle \mathbf{e}_{23}, \mathbf{e}_{125} \rangle \langle \mathbf{e}_1, \mathbf{e}_{234} \rangle$
$G_{1,7}$	$N_7 \cong (N_1)^{\circ 4}$	$\langle \mathbf{e}_{345678}, \mathbf{e}_{12347} \rangle \langle \mathbf{e}_{56}, \mathbf{e}_{3458} \rangle \langle \mathbf{e}_{34}, \mathbf{e}_{3567} \rangle \langle \mathbf{e}_2, \mathbf{e}_1 \rangle$
$G_{2,6}$	$N_8 \cong (N_1)^{\circ 3} N_2$	$\langle \mathbf{e}_{34678}, \mathbf{e}_{1258} \rangle \langle \mathbf{e}_{67}, \mathbf{e}_{346} \rangle \langle \mathbf{e}_{12}, \mathbf{e}_{15} \rangle \langle \mathbf{e}_3, \mathbf{e}_4 \rangle$
$G_{3,5}$	$N_8 \cong (N_1)^{\circ 3} N_2$	$\langle \mathbf{e}_{12456}, \mathbf{e}_{34578} \rangle \langle \mathbf{e}_{78}, \mathbf{e}_{37} \rangle \langle \mathbf{e}_{12}, \mathbf{e}_{16} \rangle \langle \mathbf{e}_4, \mathbf{e}_5 \rangle$
$G_{4,4}$	$N_7 \cong (N_1)^{\circ 4}$	$\langle \mathbf{e}_{14578}, \mathbf{e}_{12356} \rangle \langle \mathbf{e}_{78}, \mathbf{e}_{47} \rangle \langle \mathbf{e}_{23}, \mathbf{e}_{26} \rangle \langle \mathbf{e}_5, \mathbf{e}_1 \rangle$
$G_{5,3}$	$N_7 \cong (N_1)^{\circ 4}$	$\langle \mathbf{e}_{234578}, \mathbf{e}_{12367} \rangle \langle \mathbf{e}_{45}, \mathbf{e}_{48} \rangle \langle \mathbf{e}_{23}, \mathbf{e}_{27} \rangle \langle \mathbf{e}_6, \mathbf{e}_1 \rangle$
$G_{6,2}$	$N_8 \cong (N_1)^{\circ 3} N_2$	$\langle \mathbf{e}_{123456}, \mathbf{e}_{12578} \rangle \langle \mathbf{e}_{34}, \mathbf{e}_{1236} \rangle \langle \mathbf{e}_{12}, \mathbf{e}_{1345} \rangle \langle \mathbf{e}_7, \mathbf{e}_8 \rangle$
$G_{7,1}$	$N_8 \cong (N_1)^{\circ 3} N_2$	$\langle \mathbf{e}_{35678}, \mathbf{e}_{1247} \rangle \langle \mathbf{e}_{56}, \mathbf{e}_{358} \rangle \langle \mathbf{e}_{12}, \mathbf{e}_{148} \rangle \langle \mathbf{e}_8, \mathbf{e}_{123} \rangle$
$G_{8,0}$	$N_7 \cong (N_1)^{\circ 4}$	$\langle \mathbf{e}_{345678}, \mathbf{e}_{12347} \rangle \langle \mathbf{e}_{56}, \mathbf{e}_{3458} \rangle \langle \mathbf{e}_{34}, \mathbf{e}_{3567} \rangle \langle \mathbf{e}_{12}, \mathbf{e}_1 \rangle$

TABLE 15. $p+q=9$: Groups $G_{p,q}$ of order 2^{10}, $\beta = \mathbf{e}_{123456789}$

Group	Class	Group factorization
$G_{0,9}$	$S_4 \cong (N_1)^{\circ 4}C_4$	$\langle \mathbf{e}_{89}, \mathbf{e}_{2358}\rangle\langle \mathbf{e}_{67}, \mathbf{e}_{146}\rangle\langle \mathbf{e}_{23}, \mathbf{e}_{125}\rangle\langle \mathbf{e}_{1}, \mathbf{e}_{234}\rangle\langle \beta\rangle$
	$S_4 \cong (N_1)^{\circ 3}N_2C_4$	$\langle \mathbf{e}_{78}, \mathbf{e}_{1234567}\rangle\langle \mathbf{e}_{56}, \mathbf{e}_{5789}\rangle\langle \mathbf{e}_{34}, \mathbf{e}_{123}\rangle\langle \mathbf{e}_{1}, \mathbf{e}_{2}\rangle\langle \beta\rangle$
$G_{1,8}$	$\Omega_7 \cong (N_1)^{\circ 4}D_4$	$\langle \mathbf{e}_{129}, \mathbf{e}_{3479}\rangle\langle \mathbf{e}_{56}, \mathbf{e}_{3458}\rangle\langle \mathbf{e}_{34}, \mathbf{e}_{3567}\rangle\langle \mathbf{e}_{2}, \mathbf{e}_{1}\rangle\langle \beta, -\beta\rangle$
$G_{2,7}$	$S_4 \cong (N_1)^{\circ 4}C_4$	$\langle \mathbf{e}_{89}, \mathbf{e}_{245678}\rangle\langle \mathbf{e}_{67}, \mathbf{e}_{13689}\rangle\langle \mathbf{e}_{45}, \mathbf{e}_{24}\rangle\langle \mathbf{e}_{3}, \mathbf{e}_{1}\rangle\langle \beta\rangle$
	$S_4 \cong (N_1)^{\circ 3}N_2C_4$	$\langle \mathbf{e}_{89}, \mathbf{e}_{1258}\rangle\langle \mathbf{e}_{67}, \mathbf{e}_{346}\rangle\langle \mathbf{e}_{12}, \mathbf{e}_{15}\rangle\langle \mathbf{e}_{3}, \mathbf{e}_{4}\rangle\langle \beta\rangle$
$G_{3,6}$	$\Omega_8 \cong (N_1)^{\circ 3}N_2D_4$	$\langle \mathbf{e}_{3789}, \mathbf{e}_{459}\rangle\langle \mathbf{e}_{78}, \mathbf{e}_{37}\rangle\langle \mathbf{e}_{12}, \mathbf{e}_{16}\rangle\langle \mathbf{e}_{4}, \mathbf{e}_{5}\rangle\langle \beta, -\beta\rangle$
$G_{4,5}$	$S_4 \cong (N_1)^{\circ 4}C_4$	$\langle \mathbf{e}_{159}, \mathbf{e}_{2369}\rangle\langle \mathbf{e}_{78}, \mathbf{e}_{47}\rangle\langle \mathbf{e}_{23}, \mathbf{e}_{26}\rangle\langle \mathbf{e}_{5}, \mathbf{e}_{1}\rangle\langle \beta\rangle$
	$S_4 \cong (N_1)^{\circ 3}N_2C_4$	$\langle \mathbf{e}_{12567}, \mathbf{e}_{569}\rangle\langle \mathbf{e}_{34}, \mathbf{e}_{38}\rangle\langle \mathbf{e}_{12}, \mathbf{e}_{17}\rangle\langle \mathbf{e}_{5}, \mathbf{e}_{6}\rangle\langle \beta\rangle$
$G_{5,4}$	$\Omega_7 \cong (N_1)^{\circ 4}D_4$	$\langle \mathbf{e}_{169}, \mathbf{e}_{2379}\rangle\langle \mathbf{e}_{45}, \mathbf{e}_{48}\rangle\langle \mathbf{e}_{23}, \mathbf{e}_{27}\rangle\langle \mathbf{e}_{6}, \mathbf{e}_{1}\rangle\langle \beta, -\beta\rangle$
$G_{6,3}$	$S_4 \cong (N_1)^{\circ 4}C_4$	$\langle \mathbf{e}_{2368}, \mathbf{e}_{167}\rangle\langle \mathbf{e}_{45}, \mathbf{e}_{49}\rangle\langle \mathbf{e}_{23}, \mathbf{e}_{28}\rangle\langle \mathbf{e}_{7}, \mathbf{e}_{1}\rangle\langle \beta\rangle$
	$S_4 \cong (N_1)^{\circ 3}N_2C_4$	$\langle \mathbf{e}_{56}, \mathbf{e}_{123459}\rangle\langle \mathbf{e}_{34}, \mathbf{e}_{35678}\rangle\langle \mathbf{e}_{12}, \mathbf{e}_{19}\rangle\langle \mathbf{e}_{7}, \mathbf{e}_{8}\rangle\langle \beta\rangle$
$G_{7,2}$	$\Omega_8 \cong (N_1)^{\circ 3}N_2D_4$	$\langle \mathbf{e}_{789}, \mathbf{e}_{1257}\rangle\langle \mathbf{e}_{34}, \mathbf{e}_{1236}\rangle\langle \mathbf{e}_{12}, \mathbf{e}_{1345}\rangle\langle \mathbf{e}_{8}, \mathbf{e}_{9}\rangle\langle \beta, -\beta\rangle$
$G_{8,1}$	$S_4 \cong (N_1)^{\circ 4}C_4$	$\langle \mathbf{e}_{67}, \mathbf{e}_{1234569}\rangle\langle \mathbf{e}_{45}, \mathbf{e}_{4678}\rangle\langle \mathbf{e}_{23}, \mathbf{e}_{129}\rangle\langle \mathbf{e}_{9}, \mathbf{e}_{1}\rangle\langle \beta\rangle$
	$S_4 \cong (N_1)^{\circ 3}N_2C_4$	$\langle \mathbf{e}_{78}, \mathbf{e}_{1247}\rangle\langle \mathbf{e}_{56}, \mathbf{e}_{359}\rangle\langle \mathbf{e}_{12}, \mathbf{e}_{149}\rangle\langle \mathbf{e}_{9}, \mathbf{e}_{123}\rangle\langle \beta\rangle$
$G_{9,0}$	$\Omega_7 \cong (N_1)^{\circ 4}D_4$	$\langle \mathbf{e}_{129}, \mathbf{e}_{3479}\rangle\langle \mathbf{e}_{56}, \mathbf{e}_{3458}\rangle\langle \mathbf{e}_{34}, \mathbf{e}_{3567}\rangle\langle \mathbf{e}_{12}, \mathbf{e}_{1}\rangle\langle \beta, -\beta\rangle$

Appendix C. Multiplication tables for Example 7

TABLE 16. Multiplication table for $\mathbb{R}[Q_8]/\mathcal{J}$

	$1+\mathcal{J}$	$IJ+\mathcal{J}$	$I+\mathcal{J}$	$J+\mathcal{J}$
$1+\mathcal{J}$	$1+\mathcal{J}$	$IJ+\mathcal{J}$	$I+\mathcal{J}$	$J+\mathcal{J}$
$IJ+\mathcal{J}$	$IJ+\mathcal{J}$	$-1+\mathcal{J}$	$J+\mathcal{J}$	$-I+\mathcal{J}$
$I+\mathcal{J}$	$I+\mathcal{J}$	$-J+\mathcal{J}$	$-1+\mathcal{J}$	$IJ+\mathcal{J}$
$J+\mathcal{J}$	$J+\mathcal{J}$	$I+\mathcal{J}$	$-IJ+\mathcal{J}$	$-1+\mathcal{J}$

TABLE 17. Multiplication table for $(\mathbb{R}[Q_8])^{(0)}/\mathcal{J}^{(0)} \oplus (\mathbb{R}[Q_8])^{(1)}/\mathcal{J}^{(1)}$

	$1+\mathcal{J}^{(0)}$	$I+\mathcal{J}^{(0)}$	$J+\mathcal{J}^{(1)}$	$IJ+\mathcal{J}^{(1)}$
$1+\mathcal{J}^{(0)}$	$1+\mathcal{J}^{(0)}$	$I+\mathcal{J}^{(0)}$	$J+\mathcal{J}^{(1)}$	$IJ+\mathcal{J}^{(1)}$
$I+\mathcal{J}^{(0)}$	$I+\mathcal{J}^{(0)}$	$-1+\mathcal{J}^{(0)}$	$IJ+\mathcal{J}^{(1)}$	$-J+\mathcal{J}^{(1)}$
$J+\mathcal{J}^{(1)}$	$J+\mathcal{J}^{(1)}$	$-IJ+\mathcal{J}^{(1)}$	$-1+\mathcal{J}^{(0)}$	$I+\mathcal{J}^{(0)}$
$IJ+\mathcal{J}^{(1)}$	$IJ+\mathcal{J}^{(1)}$	$J+\mathcal{J}^{(1)}$	$-I+\mathcal{J}^{(0)}$	$-1+\mathcal{J}^{(0)}$

Appendix D. Multiplication tables for Q_8 and D_8

Based on the presentation of Q_8 shown in (1a), let τ be the central involution a^2. Then, $Q_8 = \{1, a, \tau, \tau a, b, ab, \tau b, \tau ab\}$. A multiplication table for Q_8 may be arranged as follows:

Table 18. A multiplication table for Q_8

	1	τ	a	τa	b	ab	τb	τab
1	1	τ	a	τa	b	ab	τb	τab
τ	τ	1	τa	a	τb	τab	b	ab
a	a	τa	τ	1	ab	τb	τab	b
τa	τa	a	1	τ	τab	b	ab	τb
b	b	τb	τab	ab	τ	a	1	τa
ab	ab	τab	b	τb	τa	τ	a	1
τb	τb	b	ab	τab	1	τa	τ	a
τab	τab	ab	τb	b	a	1	τa	τ

Table 18 exhibits a coset decomposition $Q_8 = H_1 \dot\cup H_1 b$ where H_1 is a maximal subgroup with elements $H_1 = \{1, \tau, a, \tau a\}$ while $H_1 b = \{b, \tau b, ab, \tau ab\}$. Based on this decomposition, we have a $\mathbb{Z}_2$-graded structure on the group algebra $\mathbb{R}[Q_8] = (\mathbb{R}[Q_8])^{(0)} \oplus (\mathbb{R}[Q_8])^{(1)}$ where $(\mathbb{R}[Q_8])^{(0)} = \mathrm{sp}\{h \mid h \in H_1\}$ and $(\mathbb{R}[Q_8])^{(1)} = \mathrm{sp}\{hb \mid h \in H_1\}$.

Based on the presentation of D_8 shown in (2a), let τ be the central involution a^2. Then, $D_8 = \{1, a, \tau, \tau a, b, ab, \tau b, \tau ab\}$. A multiplication table for D_8 may be arranged as follows:

Table 19. A multiplication table for D_8

	1	τ	b	τb	a	τa	ab	τab
1	1	τ	b	τb	a	τa	ab	τab
τ	τ	1	τb	b	τa	a	τab	ab
b	b	τb	1	τ	τab	ab	τa	a
τb	τb	b	τ	1	ab	τab	a	τa
a	a	τa	ab	τab	τ	1	τb	b
τa	τa	a	τab	ab	1	τ	b	τb
ab	ab	τab	a	τa	b	τb	1	τ
τab	τab	ab	τa	a	τb	b	τ	1

Table 19 exhibits a coset decomposition $D_8 = H_1 \dot\cup H_1 b$ where H_1 is a maximal subgroup with elements $H_1 = \{1, \tau, b, \tau b\}$ while $H_1 a = \{a, \tau a, ab, \tau ab\}$. Based on this decomposition, we have the $\mathbb{Z}_2$-graded structure on the group algebra $\mathbb{R}[D_8] = (\mathbb{R}[D_8])^{(0)} \oplus (\mathbb{R}[D_8])^{(1)}$ where $(\mathbb{R}[D_8])^{(0)} = \mathrm{sp}\{h \mid h \in H_1\}$ and $(\mathbb{R}[D_8])^{(1)} = \mathrm{sp}\{ha \mid h \in H_1\}$.

References

[1] R. Abłamowicz and B. Fauser: *Mathematics of CLIFFORD: A Maple package for Clifford and Grassmann algebras*, Adv. Applied Clifford Algebras 15 (2) (2005), 157–181.

[2] R. Abłamowicz and B. Fauser: `CLIFFORD`: A Maple package for Clifford and Grassmann algebras, `http://math.tntech.edu/rafal/cliff2017/`, November 2017.

[3] R. Abłamowicz and B. Fauser: *On the transposition anti-involution in real Clifford algebras I: The transposition map*, Linear and Multilinear Algebra **59** (12) (2011), 1331–1358.

[4] R. Abłamowicz and B. Fauser: *On the transposition anti-involution in real Clifford algebras II: Stabilizer groups of primitive idempotents*, Linear and Multilinear Algebra **59** (12) (2011), 1359–1381.

[5] R. Abłamowicz and B. Fauser: *On the transposition anti-involution in real Clifford algebras III: The automorphism group of the transposition scalar product on spinor spaces*, Linear and Multilinear Algebra **60** (6) (2012), 621–644.

[6] Z. Brown: *Group Extensions, Semidirect Products, and Central Products Applied to Salingaros Vee Groups Seen As* 2*-Groups*, Master Thesis, Department of Mathematics, TTU, Cookeville, TN, December 2015.

[7] V. M. Chernov, *Clifford Algebras as Projections of Group Algebras*, in *Geometric Algebra with Applications in Science and Engineering*, E. B. Corrochano and G. Sobczyk, eds., Birkhäuser, Boston (2001), 461–476.

[8] L. L. Dornhoff, *Group Representation Theory: Ordinary Representation Theory*, Marcel Dekker, Inc., New York, 1971.

[9] D. Gorenstein, *Finite Groups*, 2nd. ed., Chelsea Publishing Company, New York, 1980.

[10] J. Helmstetter: *Groupes de Clifford pour de formes quadratiques de rang quelconque.* C. R. Acad. Sci. Paris **285** (1977), 175–177.

[11] G. James and M. Liebeck, *Representations and Characters of Groups.* Cambridge University Press, 2nd ed., 2010.

[12] C. R. Leedham-Green and S. McKay, *The Structure of Groups of Prime Power Order*, Oxford University Press, Oxford, 2002.

[13] P. Lounesto: *Clifford Algebras and Spinors.* 2nd ed. Cambridge University Press, Cambridge, 2001.

[14] K. D. G. Maduranga, *Representations and Characters of Salingaros' Vee Groups*, Master Thesis, Department of Mathematics, TTU, May 2013.

[15] K. D. G. Maduranga and R. Abłamowicz: *Representations and characters of Salingaros' vee groups of low order*, Bull. Soc. Sci. Lettres Łódź Sér. Rech. Déform. **66** (1) (2016), 43–75.

[16] J. J. Rotman, *Advanced Modern Algebra*, 2nd. ed., American Mathematical Society, Providence, 2002.

[17] N. Salingaros, *Realization, extension, and classification of certain physically important groups and algebras*, J. Math. Phys. **22** (1981), 226–232.

[18] N. Salingaros, *On the classification of Clifford algebras and their relation to spinors in n dimensions*, J. Math. Phys. **23** (1) (1982), 1–7.

[19] N. Salingaros, *The relationship between finite groups and Clifford algebras*, J. Math. Phys. **25** (4) (1984), 738–742.

[20] V. V. Varlamov, *CPT Groups of Spinor Fields in de Sitter and Anti-de Sitter Spaces*, Adv. Appl. Clifford Algebras 25 (2) (2015), 487–516.

[21] A. M. Walley, *Clifford Algebras as Images of Group Algebras of Salingaros Vee Groups*, Master Thesis, Department of Mathematics, TTU, Cookeville, TN, May 2017.

Rafał Abłamowicz
Department of Mathematics
Tennessee Technological University
Cookeville, TN 38505, U.S.A.
e-mail: `rablamowicz@tntech.edu`

Manisha Varahagiri
Department of Mathematics
Tennessee Technological University
Cookeville, TN 38505, U.S.A.
e-mail: `vvarahagi42@students.tntech.edu`

Anne Marie Walley
Department of Mathematics
Tennessee Technological University
Cookeville, TN 38505, U.S.A.
e-mail: `amparke42@students.tntech.edu`