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SPINOR MODULES OF CLIFFORD ALGEBRAS IN CLASSES N_{2k-1} AND Ω_{2k-1} ARE DETERMINED BY IRREDUCIBLE NONLINEAR CHARACTERS OF CORRESPONDING SALINGAROS VEE GROUPS

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Spinor Modules of Clifford Algebras in Classes N_{2k-1} and Ω_{2k-1} are Determined by Irreducible Nonlinear Characters of Corresponding Salingaros Vee Groups

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Abstract. Clifford algebras $C\ell_{2,0}$ and $C\ell_{1,1}$ are isomorphic simple algebras whose Salingaros vee groups belong to a class N_1. The algebras are isomorphic to the quotient algebra $\mathbb{R}[D_8]/\mathcal{J}$ of the group algebra of the dihedral group D_8 modulo an ideal $\mathcal{J} = (1+\tau)$ where τ is a central involution in D_8. Since all irreducible characters of D_8, including a single nonlinear character of degree 2, can be realized over $\mathbb{R}$, spinor representations of the Clifford algebras can be realized over $\mathbb{R}$ and so $C\ell_{2,0} \cong C\ell_{1,1} \cong \mathbb{R}(2)$. Spinor modules in $C\ell_{2,0}$ and $C\ell_{1,1}$ are isomorphic to irreducible $\mathbb{R}D_8$-submodules of dimension 2 of the regular module $\mathbb{R}D_8$. As such, they are uniquely determined by the nonlinear character of degree 2. These results are generalized to the vee groups in classes N_{2k-1} and Ω_{2k-1} $(1 \le k \le 4)$. It is proven that each irreducible character of $G_{p,q}$ in these classes can be realized over $\mathbb{R}$. Consequently, every nonlinear character of $G_{p,q}$ uniquely determines a spinor module of $C\ell_{p,q}$ which is faithful (resp. unfaithful) when $G_{p,q}$ is in the class N_{2k-1} (resp. Ω_{2k-1}). This paper is a continuation of [1].

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1. Introduction

In a series of papers [11–13], Salingaros defined five families N_{2k-1}, N_{2k}, Ω_{2k-1}, Ω_{2k}, S_k $(k \ge 1)$ of finite 2-groups contained in Clifford algebras $C\ell_{p,q}$. We refer to these groups as *Salingaros vee groups* and we denote them as $G_{p,q}$

when $G_{p,q} \subset C\ell_{p,q}$. Abłamowicz *et al.* have shown in [1] that each Clifford algebra $C\ell_{p,q}$ is an image of a group algebra $\mathbb{R}[G_{p,q}]$ and, as a consequence, Clifford algebras $C\ell_{p,q}$ can be classified in terms of the Salingaros vee groups into five isomorphism classes in accordance with the well-known Periodicity of Eight [8].

The approach to Clifford algebras presented in [1] opened a possibility of relating properties of $C\ell_{p,q}$ and properties of its algebraic substructures, such as its spinor modules (minimal one-sided ideals), to the properties of the group algebra $\mathbb{R}[G_{p,q}]$ and its regular $\mathbb{R}G_{p,q}$-module [6]. In particular, these properties can be related to irreducible characters of the group $G_{p,q}$.

The main goal for this paper is to show how the nonlinear irreducible character(s) of the vee groups in the class N_{2k-1} (resp. Ω_{2k-1}) determine(s) the spinor modules, and hence, spinor representations of simple (resp. semisimple) Clifford algebras $C\ell_{p,q} \cong \mathbb{R}(2^l)$ (resp. $C\ell_{p,q} \cong {}^2\mathbb{R}(2^{l-1})$) for $p - q = 0, 2 \pmod 8$ (resp. $p - q = 1 \pmod 8$).

Recall that for $k \geq 1$, each group in the family N_{2k-1} is an extra-special central product[1] $(D_8)^{\circ k}$ of k copies of the dihedral group D_8 whereas each group in the family Ω_{2k-1} is a central product $(D_8)^{\circ k} \circ (C_2 \times C_2)$ where $\circ$ denotes the central product and C_2 denotes a cyclic group of order 2 [1,11–13].

The first step in accomplishing our goal is to show that each nonlinear character χ of $G_{p,q}$ can be realized over $\mathbb{R}$. We do that by using the Frobenius-Schur Count of Involutions to show that an indicator function $\iota\chi = 1$ for every irreducible character of $G_{p,q}$ in either class (Section 4). Then, for any group $G_{p,q}$ in N_{2k-1}, we relate faithfulness of its single nonlinear character to the simplicity of the corresponding Clifford algebra $C\ell_{p,q}$ and faithfulness of its spinor representation. Since any group $G_{p,q}$ in Ω_{2k-1} has two unfaithful nonlinear irreducible characters, this explains why the corresponding Clifford algebras $C\ell_{p,q}$ are semisimple and why their spinor representations are not faithful (cf. [8, Sect. 17.6]).

The paper is organized as follows.

Section 2 recalls a known fact that all characters of D_8 from the class N_1 can be realized over $\mathbb{R}$ [6]. As a consequence, since $\mathbb{R}[D_8]/\mathcal{J} \cong C\ell_{2,0} \cong C\ell_{1,1}$ (cf. [1]), the spinor representation of these Clifford algebras is equivalent to a faithful irreducible representation of D_8 with a character of degree 2.

Section 3 illustrates in greater detail how the single nonlinear irreducible character χ_5 of D_8 defines the spinor modules of $C\ell_{2,0} \cong C\ell_{1,1}$. First, the regular module $\mathbb{R}D_8$ is decomposed into a direct sum of irreducible $\mathbb{R}D_8$-submodules (c.f. [6, Maschke Theorem]) related to the irreducible characters of D_8. This is accomplished via the well-known formula (13). We discuss the irreducible $\mathbb{R}D_8$-submodule of the subalgebra $\mathbb{R}[D_8]\theta_2 \cong C\ell_{2,0} \cong C\ell_{1,1}$ of the group algebra $\mathbb{R}[D_8]$ (θ_2 is a central idempotent in $\mathbb{R}[D_8]$) whose character is the nonlinear faithful character of D_8. We prove Thm. 2 saying that the spinor

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[1] For all definitions and properties of extra-special p-groups, see [5, 7]. See also [1] and references therein.

modules of $C\ell_{2,0}$ and $C\ell_{1,1}$ are isomorphic to that irreducible submodule, and so they are determined by χ_5.

In Section 4 we generalize to Salingaros vee groups in classes N_{2k-1} and Ω_{2k-1} for $1 \leq k \leq 4$. We recall the definition and properties of the indicator function and state the Frobenius-Schur Count of Involutions [6]. For each group in the said classes we give the order structure of that group and prove our first main result (Thm. 4) that each irreducible character of these groups can be realized over $\mathbb{R}$. This theorem gives a foundation to our second main result (at least for $1 \leq k \leq 4$) stated as Thm. 5 that when $G_{p,q}$ is in the class N_{2k-1} (resp. Ω_{2k-1}), the spinor representation is faithful (resp. non faithful) and it is determined by one of the nonlinear irreducible characters χ of the group. We find a relation between the degree $\chi(1)$ of such character, the Radon-Hurwitz number r_{q-p}, and the signature (p, q). We sketch a proof of that theorem and illustrate it on $\mathbb{R}[G_{2,1}]/\mathcal{J} \cong C\ell_{2,1}$, which is a semi-simple algebra in the class Ω_1. Using the character table of $G_{2,1}$, we show how two distinct nonlinear non faithful irreducible characters of degree 2 determine, in the end, a faithful yet reducible double spinor module for $C\ell_{2,1}$.

In the end, we conjecture that our results extend to the vee groups in all classes N_{2k-1} and Ω_{2k-1} for $k \geq 1$, that is, to all Clifford algebras $C\ell_{p,q}$ for $p - q = 0, 1, 2 \pmod 8$.

2. Nonlinear character of D_8 can be realized over $\mathbb{R}$

We recall a few definitions and results from representation theory of finite groups. Our main reference is [6].

Definition 1. *An element g of a finite group G is said to be real if g is conjugate to g^{-1}. If g is real, then the conjugacy class g^G is said to be real. A character χ of G is real if $\chi(g)$ is real for all $g \in G$.*

For example, the conjugacy class $\{1\}$ of the identity element of G is real, and the trivial character 1_G of G is real.

Theorem 1. *The number of real irreducible characters of G is equal to the number of real conjugacy classes of G.*

Corollary 1. *The group G has a nontrivial real irreducible character if and only if the order of G is even.*

Definition 2. *Let χ be a character of the group G. We say that χ can be realized over $\mathbb{R}$ if there is a representation $\rho : G \to \mathrm{GL}(n, \mathbb{C})$ with character χ, such that all the entries in each matrix $g\rho$ ($g \in G$) are real.*

Every character χ which can be realized over $\mathbb{R}$ is a real character, but the converse is false [6, Example 23.3(2)].

Let $G = D_8 = \langle a, b \mid a^4 = b^2 = 1, b^{-1}ab = a^{-1} \rangle$. Thus,

$$D_8 = \{1, a^2, a, a^3, b, a^2b, ab, a^3b\}, \tag{1}$$

Table 1. Character table for D_8 in class N_1

g_i	1	a^2	a	b	ab
$\lvert C_G(g_i)\rvert$	8	8	4	4	4
χ_1	1	1	1	1	1
χ_2	1	1	1	−1	−1
χ_3	1	1	−1	1	−1
χ_4	1	1	−1	−1	1
χ_5	2	−2	0	0	0

the order structure of D_8 is $[1,5,2]$ with $|a| = |a^3| = 4$ and the center $Z(G) = \{1, a^2\}$. A character table of G is given in Table 1 where g_i is a representative of the conjugacy class K_i and $|C_G(g_i)|$ is the order of the centralizer $C_G(g_i)$ of g_i in G for $1 \leq i \leq 5$. Recall that the conjugacy classes of D_8 are as follows:

$$K_1 = \{1\},\ \ K_2 = \{a^2\},\ \ K_3 = \{a, a^3\},\ \ K_4 = \{b, a^2 b\},\ \ K_5 = \{ab, a^3 b\} \quad (2)$$

and $|K_i| = [G : C_G(g_i)] = |G|/|C_G(g_i)|$. Thus, every element of D_8 is real and so each conjugacy class is real as well. Each character of D_8 can be realized over $\mathbb{R}$. In particular, the irreducible character of degree 2 can be realized over $\mathbb{R}$, since

$$a\rho = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad b\rho = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad (3)$$

provides a representation ρ of G with character χ_5 such that all the matrices $g\rho$ $(g \in G)$ have real entries.[2]

From Table 1 we gather that $\ker \chi_5 = \{1\}$, that is, χ_5 is faithful so the representation ρ is faithful as well.

The following are two important consequences of the defining relations on the generators a and b of D_8:

1. The elements b and ab are of order 2 and $(ab)b = a^2(b(ab))$.
2. The element a is of order 4 and $ba = a^3 b = a^2(ab)$.

Recall from [1] that Clifford algebras $C\ell_{2,0}$ and $C\ell_{1,1}$ are images of the group algebra $\mathbb{R}[D_8]$ where the surjective algebra maps from $\mathbb{R}[D_8]$ onto each Clifford algebra have kernel equal to $\mathcal{J} = (1 + a^2)$. Thus,

$$\mathbb{R}[D_8]/\mathcal{J} \cong C\ell_{2,0} \cong C\ell_{1,1}. \quad (4)$$

In view of the first consequence, the first isomorphism in (4) factors a surjective $\mathbb{R}$-algebra map

$$\psi_1 : \mathbb{R}[D_8] \to C\ell_{2,0} : \quad 1 \mapsto 1,\ \ b \mapsto \mathbf{e}_1,\ \ ab \mapsto \mathbf{e}_2 \quad (5)$$

[2] We are using a notational convention from [6] where morphisms are written on the right.

where $\mathbf{e}_1, \mathbf{e}_2$ provide an orthonormal basis in the Euclidean space $\mathbb{R}^2$, hence, $\mathbf{e}_1$ and $\mathbf{e}_2$ anticommute and square to 1 in $C\ell_{2,0}$. Notice that $(ab)\rho = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and

$$(b\rho)(b\rho) = 1,\ (ab)\rho(ab)\rho = 1,\ (bab)\rho + (abb)\rho = 0. \tag{6}$$

Thus, the following matrices may be chosen to represent the 1-vector generators of $C\ell_{2,0}$:

$$\mathbf{e}_1 \mapsto \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{e}_2 \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{7}$$

which we recognize as a spinorial (faithful) representation of $C\ell_{2,0}$. [3]

The second consequence allows us to recover a spinor representation of $C\ell_{1,1}$. We first define a surjective algebra map

$$\psi_2 : \mathbb{R}[D_8] \to C\ell_{1,1} : \quad 1 \mapsto 1,\ \ b \mapsto \mathbf{e}_1,\ \ a \mapsto \mathbf{e}_2 \tag{8}$$

and find that the following matrices represent the 1-vector generators of $C\ell_{1,1}$:

$$\mathbf{e}_1 \mapsto \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{e}_2 \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{9}$$

which we again recognize as a spinor representation of $C\ell_{1,1}$. [4]

We summarize our findings.

1. A single faithful irreducible real representation of D_8 with character of degree 2 provides the spinor representation of Clifford algebras $C\ell_{2,0}$ and $C\ell_{1,1}$. This explains why $C\ell_{2,0} \cong C\ell_{1,1} \cong \mathrm{Mat}(2,\mathbb{R})$.
2. Spinor representations of $C\ell_{2,0}$ and $C\ell_{1,1}$ are real because Salingaros vee groups $G_{2,0}$ and $G_{1,1}$ belong to the same isomorphism class N_1, that is, $D_8 \cong G_{2,0} \cong G_{1,1}$, or, equivalently, because $\mathbb{R}[D_8]/\mathcal{J} \cong C\ell_{2,0} \cong C\ell_{1,1}$.
3. The representations (7) and (9) are faithful and equivalent. They are also traceless because they are defined in terms of images of elements b, ab, and a which belong to the conjugacy classes of D_8 on which the character χ_5 is zero.
4. Since the maps (5) and (8) are defined modulo the nontrivial central element a^2, which maps into the nontrivial central element -1 in either Clifford algebra, the spinor representations (7) and (9) are defined modulo $-I_2$ for a 2×2 identity matrix I_2.
5. The maps (5) and (8) are defined so that when considered as group isomorphisms from D_8 into $G_{2,0} \subset C\ell_{2,0}^*$ and $G_{1,1} \subset C\ell_{1,1}^*$, respectively, the orders of the image and preimage are equal.
6. Since the only two elements in D_8 of order 4, namely, a and a^3, commute, it is impossible to construct $C\ell_{0,2} \cong \mathbb{H}$ as an image of $\mathbb{R}[D_8]$. In fact, one must use the quaternion group Q_8 instead of D_8 so that $\mathbb{R}[Q_8]/\mathcal{J} \cong C\ell_{0,2}$ as shown in [1].[5]

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[3] This representation is equivalent to the one shown in [2, 3, 8].

[4] This representation is equivalent to the one shown in [2–4].

[5] Recall that although the character table of Q_8 is identical to Table 1, and so the character χ_5 of Q_8 is real, χ_5 cannot be realized over $\mathbb{R}$ [6, Example 23.18(3)]. This is the reason

3. Nonlinear character of D_8 in spinor modules of $C\ell_{2,0}$ and $C\ell_{1,1}$

3.1. Algebra $\mathbb{R}[D_8]\theta_2$ as determined by the nonlinear character of D_8

As in the previous section, we let $G = D_8$ and $\tau = a^2 \in Z(G)$. Let $\theta_1 = \frac{1}{2}(1+\tau)$ and $\theta_2 = \frac{1}{2}(1-\tau)$ in $\mathbb{R}[G]$. Therefore, θ_1 and θ_2 are central mutually annihilating idempotents adding up to 1 in $\mathbb{R}[G]$. Hence,

$$\mathbb{R}[G] = \mathbb{R}[G]\theta_2 \oplus \mathbb{R}[G]\theta_1 = \mathbb{R}[G]\theta_2 \oplus \mathcal{J} \tag{10}$$

where $\mathcal{J} = \mathbb{R}[G]\theta_1 = (1+\tau)$ as in Sect. 2. Thus, $\mathbb{R}[G]\theta_1$ and $\mathbb{R}[G]\theta_2$ are two subalgebras of $\mathbb{R}[G]$ of dimension four each. Therefore, $\mathbb{R}[G]/\mathcal{J} \cong \mathbb{R}[G]\theta_2$ as $\mathbb{R}$-algebras and we have the following commutative diagram:

$$\begin{array}{ccc} \mathbb{R}[G] & \xrightarrow{\ \Psi\ } & \mathbb{R}[G]\theta_2 \\ {\scriptstyle\pi}\downarrow & \nearrow_{\varphi} & \\ \mathbb{R}[G]/\mathcal{J} & & \end{array}\quad , \qquad \begin{array}{ccc} u & \stackrel{\Psi}{\longmapsto} & u\theta_2 \\ {\scriptstyle\pi}\downarrow & \nearrow_{\varphi} & \\ u+\mathcal{J} & & \end{array}\quad , \tag{11}$$

where $u + \mathcal{J} = u\theta_2 + (u\theta_1 + \mathcal{J})$. So, $uv + \mathcal{J} \stackrel{1:1}{\longleftrightarrow} uv\theta_2$ for $u, v \in \mathbb{R}[G]$. Thus, rather than performing computations in the quotient algebra $\mathbb{R}[G]/\mathcal{J}$, it is more convenient to compute in the isomorphic algebra $\mathbb{R}[G]\theta_2$.

Let U_i be an irreducible $\mathbb{R}G$-submodule of a regular module $\mathbb{R}G$ with character χ_i, $1 \le i \le 5$, shown in Table 1. Then,

$$\mathbb{R}G = U_1 \oplus U_2 \oplus U_3 \oplus U_4 \oplus U_5 \oplus U_5' \tag{12}$$

where U_5' is $\mathbb{R}D_8$-isomorphic to U_5, and so χ_5 is the character of U_5'. Suppose that $\mathbb{R}G = W_1 \oplus W_2$, where W_1 and W_2 are $\mathbb{R}G$-submodules which have no common composition factor[6]. Write $1 = e_1 + e_2$ where $e_1 \in W_1$ and $e_2 \in W_2$. Let χ be the character of W_1. Then,

$$e_1 = \frac{1}{|G|} \sum_{g \in G} \chi(g^{-1}) g, \tag{13}$$

and similarly for e_2. It can be easily shown that e_1 and e_2 are mutually annihilating idempotents [6].

Let us now repeatedly apply these ideas to the regular module $\mathbb{R}G$ whose complete decomposition into the irreducibles is shown in (12). This way, we can compute four (primitive) idempotents e_1, e_2, e_3, e_4 using the four irreducible characters χ_1, χ_2, χ_3, χ_4, and formula (13). Let $W_1 = U_1 \oplus U_2 \oplus U_3 \oplus U_4$ and $W_2 = U_5 \oplus U_5'$. Then, W_1 and W_2 do not have a common

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why $G_{0,2}$ belongs to Salingaros class N_2 and so $C\ell_{0,2}$ does not have a real 2×2 spinor representation.

[6]Let G be a finite group. If V is a $\mathbb{C}G$-module and U is an irreducible $\mathbb{C}G$-submodule, then we say that U is a *composition factor* of V if V has a $\mathbb{C}G$-submodule which is $\mathbb{C}G$-isomorphic to U. Two $\mathbb{C}G$-modules V and W are said to have a *common composition factor* if there is an irreducible $\mathbb{C}G$-module which is a composition factor of V and W [6]. Since all characters of D_8 can be realized over $\mathbb{R}$, we apply these definitions to the real regular module $\mathbb{R}D_8$.

composition factor and the character χ of W_2 is $2\chi_5$. Let e_5 be an idempotent defined by χ. Then,

$$e_1 + e_2 + e_3 + e_4 = \frac{1}{2}(1 + a^2) = \theta_1, \tag{14}$$

$$e_5 = \frac{1}{8}\sum_{g\in G} 2\chi_5(g^{-1})g = \frac{1}{2}(1 - a^2) = \theta_2, \tag{15}$$

Thus,

$$\mathcal{J} = (1 + a^2) = (\theta_1) = \mathbb{R}[G]\theta_1 = \mathrm{sp}\{e_1, e_2, e_3, e_4\}, \tag{16}$$

$$\mathbb{R}[G]\theta_2 = \{ue_5 \mid u \in \mathbb{R}[G]\} = \mathrm{sp}\{q_1, q_2, q_3, q_4\} \tag{17}$$

where $q_1 = e_5, q_2 = ae_5, q_3 = be_5, q_4 = abe_5$. Furthermore,

$$\mathbb{R}[G]\theta_2 \cong \mathbb{R}[G]/\mathcal{J} = \mathrm{sp}\{1 + \mathcal{J}, a + \mathcal{J}, b + \mathcal{J}, ab + \mathcal{J}\}. \tag{18}$$

The following is a multiplication table for the basis elements of $\mathbb{R}[G]\theta_2$:

TABLE 2. Multiplication table in $\mathbb{R}[G]\theta_2 \cong C\ell_{1,1} \cong C\ell_{2,0}$

	q_1	q_2	q_3	q_4
q_1	q_1	q_2	q_3	q_4
q_2	q_2	$-q_1$	q_4	$-q_3$
q_3	q_3	$-q_4$	q_1	$-q_2$
q_4	q_4	q_3	q_2	q_1

Thus, q_1 is the identity of the algebra $\mathbb{R}[G]\theta_2$, and $|q_2| = 4$, $|q_3| = |q_4| = 2$ in the multiplicative group $(\mathbb{R}[G]\theta_2)^*$. As expected, the algebra $\mathbb{R}[G]\theta_2$ is $\mathbb{R}$-isomorphic[7] to $C\ell_{2,0}$ and $C\ell_{1,1}$. The isomorphisms may be defined on the basis elements as

$$\varphi_1 : \mathbb{R}[G]\theta_2 \to C\ell_{2,0} \quad \text{with} \quad q_1 \mapsto 1,\ q_2 \mapsto \mathbf{e}_{12},\ q_3 \mapsto \mathbf{e}_2,\ q_4 \mapsto \mathbf{e}_1, \tag{19}$$

$$\varphi_2 : \mathbb{R}[G]\theta_2 \to C\ell_{1,1} \quad \text{with} \quad q_1 \mapsto 1,\ q_2 \mapsto \mathbf{e}_2,\ q_3 \mapsto \mathbf{e}_{12},\ q_4 \mapsto \mathbf{e}_1, \tag{20}$$

and then extended by linearity to any element in the domain. Notice that when defining each map, care has to be taken to preserve the orders of the group elements in $(\mathbb{R}[G]\theta_2)^*$ and of their images in Salingaros groups $G_{2,0}$ and $G_{1,1}$, respectively.

3.2. Spinor modules of $C\ell_{2,0}$ and $C\ell_{1,1}$ as determined by the nonlinear character of D_8

3.2.1. Spinor module of $C\ell_{2,0}$. Recall that element $f_1 = \frac{1}{2}(1+\mathbf{e}_1)$ is a primitive idempotent in $C\ell_{2,0}$ and so S_1 defined as $C\ell_{2,0}f_1 = \mathrm{sp}\{f_1, \mathbf{e}_2 f_1\}$ is a minimal left ideal which may serve as a two-dimensional spinor space of the algebra. At the same time, S_1 is a right $\mathbb{K}$-module where $\mathbb{K} = f_1 C\ell_{2,0} f_1 \cong \mathbb{R}$. Define $\tilde{f}_1 = \varphi_1^{-1}(f_1) = \frac{1}{2}(q_1 + q_4)$ where φ_1 is the isomorphism $\mathbb{R}[G]\theta_2 \to C\ell_{2,0}$

[7] In fact, it is $\mathbb{Z}_2$-isomorphic to exactly one of $C\ell_{2,0}$ and $C\ell_{1,1}$ as shown in [1].

defined in (19). Likewise, $f_2 = \frac{1}{2}(1 - \mathbf{e}_1)$ is a primitive idempotent in $C\ell_{2,0}$ orthogonal to f_1 and let $\tilde{f}_2 = \varphi^{-1}(f_2)$. Let $S_2 = C\ell_{2,0}f_2$. Then, $\tilde{f}_1$ and $\tilde{f}_2$ are mutually annihilating primitive idempotents in $\mathbb{R}[G]\theta_2$ so that

$$\begin{aligned}\mathbb{R}[G]\theta_2 &= \mathbb{R}[G]\theta_2\tilde{f}_1 \oplus \mathbb{R}[G]\theta_2\tilde{f}_2 = \mathbb{R}[G]\tilde{f}_1 \oplus \mathbb{R}[G]\tilde{f}_2 \\ &= \mathrm{sp}\{\tilde{f}_1, q_2\tilde{f}_1\} \oplus \mathrm{sp}\{\tilde{f}_2, q_2\tilde{f}_2\}. \end{aligned} \tag{21}$$

since $\theta_2 = \tilde{f}_1 + \tilde{f}_2$, $\theta_2\tilde{f}_1 = \tilde{f}_1$, and $\theta_2\tilde{f}_2 = \tilde{f}_2$.

Proposition 1. *Let $G = D_8$, and, let $\tilde{S}_1 = \mathrm{sp}\{\tilde{f}_1, q_2\tilde{f}_1\}$ and $\tilde{S}_2 = \mathrm{sp}\{\tilde{f}_2, q_2\tilde{f}_2\}$ be the left $\mathbb{R}[G]$-modules defined above. When viewed as left $\mathbb{R}G$-modules, each module has character χ_5 of G. Thus, they are isomorphic to the irreducible $\mathbb{R}G$-submodule U_5 of the regular module $\mathbb{R}G$.*

Proof. We will only show that the character of $\tilde{S}_1$ is χ_5 by defining a (left) representation ρ of G in $\tilde{S}_1$, that is, $\rho : G \to \mathrm{GL}(2,\mathbb{R})$ is group homomorphism. Then, we find that the generators of G are mapped as follows:

$$a \mapsto A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad b \mapsto B = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \tag{22}$$

which gives:

TABLE 3. Representation ρ and its character χ

g	1	a^2	a	a^3
$g\rho$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$
$\chi(g)$	2	-2	0	0

g	b	a^2b	ab	a^3b
$g\rho$	$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$
$\chi(g)$	0	0	0	0

It can be seen that $\ker\chi = \{1\}$, thus, ρ is a faithful representation of G since $A^4 = B^2 = I_2$, and $B^{-1}AB = A^{-1}$. Furthermore, $\chi = \chi_5$.

In a similar way one can show that the character of $\tilde{S}_2$ is χ_5 as well. □

Under the isomorphism $\varphi_1 : \mathbb{R}[D_8]\theta_2 \to C\ell_{2,0}$ from (19), we have $q_4 \mapsto \mathbf{e}_1$ and $q_3 \mapsto \mathbf{e}_2$. Thus, the generators $\mathbf{e}_1$ and $\mathbf{e}_2$ of $C\ell_{2,0}$ are assigned the following matrices through the representation ρ :

$$\mathbf{e}_1 \mapsto E_1 = q_4\rho = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad \mathbf{e}_2 \mapsto E_2 = q_3\rho = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}. \tag{23}$$

Matrices E_1 and E_2 fulfill expected relations: $E_1^2 = E_2^2 = I_2$ and $E_1E_2 + E_2E_1 = 0$.

Likewise, under the isomorphism $\varphi_2 : \mathbb{R}[D_8]\theta_2 \to C\ell_{1,1}$ from (20), we have $q_4 \mapsto \mathbf{e}_1$ and $q_2 \mapsto \mathbf{e}_2$. Thus, the generators $\mathbf{e}_1$ and $\mathbf{e}_2$ of $C\ell_{1,1}$ are assigned the following matrices by ρ :

$$\mathbf{e}_1 \mapsto E_1 = q_4\rho = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad \mathbf{e}_2 \mapsto E_2 = q_2\rho = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{24}$$

This time, matrices E_1 and E_2 fulfill these relations: $E_1^2 = I_2$, $E_2^2 = -I_2$ and $E_1E_2 + E_2E_1 = 0$.

Theorem 2. *The spinor modules in $C\ell_{2,0}$ and $C\ell_{1,1}$ represented as minimal left ideals are isomorphic to an irreducible $\mathbb{R}G$-submodule of $\mathbb{R}[D_8]\theta_2$, namely, $\mathbb{R}[D_8]\tilde{f}_1$. Thus, they are uniquely determined by the irreducible character χ_5 of degree 2 of D_8. In particular, 2×2 matrices representing generators $\mathbf{e}_1$ and $\mathbf{e}_2$ of $C\ell_{2,0}$ and $C\ell_{1,1}$ are traceless because they represent non-central elements of D_8 which belong to conjugacy classes on which the character χ_5 is zero.*

4. A generalization to classes N_{2k-1} and Ω_{2k-1}

The goal for this section is to generalize previous results to Clifford algebras $C\ell_{p,q}$ related to Salingaros classes N_{2k-1} and Ω_{2k-1} for $p - q = 0, 2 \pmod 8$ and $p - q = 1 \pmod 8$, respectively [1]. We recall from [6] a definition of an indicator function, its properties, and the so called *Frobenius-Schur Count of Involutions.* We will use the latter to establish our main results for this section.

Let V be a $\mathbb{C}G$-module with character χ. Then, χ^2 is the character of the $\mathbb{C}G$-module $V \otimes V$, and $\chi^2 = \chi_S + \chi_A$ where χ_S (resp. χ_A) is the character of the symmetric (resp. antisymmetric) part of $V \otimes V$. Recall that

$$\chi_S(g) = \frac{1}{2}(\chi^2(g) + \chi(g^2)) \ \text{ and } \ \chi_A(g) = \frac{1}{2}(\chi^2(g) - \chi(g^2)) \quad (g \in G). \tag{25}$$

Recall that 1_G denotes the trivial character of G.

Definition 3. *If χ is an irreducible character of a group G, then we define an indicator $\iota\chi$ by*

$$\iota\chi = \begin{cases} 0, & \text{if } 1_G \text{ is not a constituent of } \chi_S \text{ or } \chi_A, \\ 1, & \text{if } 1_G \text{ is a constituent of } \chi_S, \\ -1 & \text{if } 1_G \text{ is a constituent of } \chi_A. \end{cases} \tag{26}$$

We call ι the indicator function on the set of irreducible characters of G.[8]

Note that $\iota\chi \neq 0$ if an only if χ is real. The following result [6] relates the indicator function to the structure of G.

[8] An irreducible character ψ is a *constituent* of a character χ if $\langle \chi, \psi \rangle \neq 0$, where $\langle \cdot, \cdot \rangle$ is an inner product on the characters of a group G defined as $\langle \chi, \psi \rangle = \frac{1}{|G|} \sum_g \chi(g)\psi(g^{-1}) = \sum_{i=1}^{N} \frac{\chi(g_i)\psi(g_i^{-1})}{|C_G(g_i)|}$ where the summation is over all conjugacy classes K_i with representatives g_i and $|C_G(g_i)|$ denotes the order of the centralizer of g_i in G.

Theorem 3. *For all $x \in G$,*

$$\sum_{\chi} (\iota\chi)\chi(x) = |\{y \in G \mid y^2 = x\}|, \tag{27}$$

where the sum is taken over all irreducible characters χ of G.

A proof of the following proposition can be found in [6].

Proposition 2 (The Frobenius-Schur Count of Involutions). *For each irreducible character χ of G, we have*

$$\iota\chi = \begin{cases} 0, & \text{if } \chi \text{ is not real,} \\ 1, & \text{if } \chi \text{ can be realized over } \mathbb{R}, \\ -1 & \text{if } \chi \text{ is real, but } \chi \text{ cannot be realized over } \mathbb{R}. \end{cases} \tag{28}$$

Moreover,

$$\sum_{\chi} (\iota\chi)\chi(1) = |\{y \in G \mid y^2 = 1\}| = 1 + t, \tag{29}$$

where t is equal to the number of involutions in G.

Example 1. *It can be shown by directly computing χ_S and χ_A for each irreducible character χ of D_8 that 1_{D_8} is a constituent of χ_S. Thus, $\iota\chi = 1$ for each χ, and so each irreducible character of D_8 can be realized over $\mathbb{R}$. A faster way to find all values $\iota\chi$ is to count the involutions: since D_8 has $t = 5$ involutions,*

$$\sum_{\chi} (\iota\chi)\chi(1) = 6, \tag{30}$$

which implies that $\iota\chi = 1$ for each character χ. This leads to Thm. 2 from the previous section.

Example 2. *A character table for the quaternion group*

$$Q_8 = \langle a, b \mid a^4 = 1, a^2 = b^2, b^{-1}ab = a^{-1} \rangle \tag{31}$$

is the same as the character table for D_8 given as Table 1. However, the order structure of Q_8 is $[1, 1, 6]$ with the central element a^2 being the only involution in Q_8. Direct computation or using the involution count, gives $\iota\chi_i = 1$ for $1 \le i \le 4$ and $\iota\chi_5 = -1$ since 1_{Q_8} is a constituent of the antisymmetric part of χ_5^2. Hence, the character χ_5 cannot be realized over the reals. This explains why Clifford algebra $C\ell_{0,2} \cong \mathbb{H}$ does not have a 2×2 real representation: instead, it does have a 2×2 complex representation [8, Sect. 5.6].

In Tables 4 and 5 we have collected information about representative Salingaros vee groups $G_{p,q}$ in classes N_{2k-1} and Ω_{2k-1} $(1 \le k \le 4)$ for $p - q = 0, 2 \,(\mathrm{mod}\, 8)$ and $p - q = 1 \,(\mathrm{mod}\, 8)$, respectively. From Sect. 2 we recall that the corresponding Clifford algebras $C\ell_{p,q}$ are isomorphic to the quotient algebras $\mathbb{R}[G_{p,q}]/\mathcal{J}$ with groups in the N_{2k-1} (resp. Ω_{2k-1}) class yielding simple (resp. semisimple) algebras. These algebras have real spinor representations since $p - q = 0, 1, 2 \,(\mathrm{mod}\, 8)$. Following [8,9], our notation is as follows:

- $\mathbb{R}(2^l)$ denotes the isomorphism class $\mathrm{Mat}(2^l,\mathbb{R})$ while ${}^2\mathbb{R}(2^{l-1})$ denotes the isomorphism class $\mathrm{Mat}(2^{l-1},\mathbb{R}) \oplus \mathrm{Mat}(2^{l-1},\mathbb{R})$ of $C\ell_{p,q}$ where $l = q - r_{q-p}$ and r_i is Radon-Hurwitz number,[9]
- C.O.S. and G.O.S. denote, respectively, the order structure of the group center $Z(G_{p,q})$ and the group $G_{p,q}$ itself,
- $M = 2^{p+q}$ is the number of linear characters of $G_{p,q}$ when $p+q \geq 2$,
- N is the number of conjugacy classes in $G_{p,q}$ equal to $1 + 2^{p+q}$, when $p+q$ is even, and $2 + 2^{p+q}$, when $p+q$ is odd,
- since $G'_{p,q} = \{\pm 1\}$, every conjugacy class $g^{G_{p,q}}$ contains exactly one element when $g \in Z(G_{p,q})$, or, two elements when $g \notin Z(G_{p,q})$,
- since $Z(G_{p,q}) \cong C_2$ when $p+q$ is even, and $Z(G_{p,q}) \cong C_2 \times C_2$ when $p+q$ is odd, groups in classes N_{2k-1} have two singleton conjugacy classes while groups in classes Ω_{2k-1} have four singleton conjugacy classes, and all other classes have two elements each of the same order two or four,
- $L = N - M$ is the number of nonlinear characters of $G_{p,q}$,
- t is the number of involutions in $G_{p,q}$.

The order structure of each vee group has been computed directly with `CLIFFORD` [2, 3]. We know that each class contains usually more than one group $G_{p,q}$, which results in having different yet isomorphic Clifford algebras $C\ell_{p,q}$ in each class. For example, when $p+q=2$, we have $G_{1,1} \cong G_{2,0}$ and so $C\ell_{1,1} \cong C\ell_{2,0} \cong \mathbb{R}(2)$ or, when $p+q=7$, we have $G_{0,7} \cong G_{4,3}$ and so $C\ell_{0,7} \cong C\ell_{4,3} \cong {}^2\mathbb{R}(8)$ in accordance with the Periodicity of Eight [8].

Table 4. Classes N_{2k-1} for $1 \leq k \leq 4$ and $p - q = 0, 2 \pmod 8$

Class	Group	$C\ell_{p,q}$	Center	C.O.S.	G.O.S.	L	M	N	t
N_1	$G_{1,1}$	$\mathbb{R}(2)$	C_2	$[1,1,0]$	$[1,5,2]$	1	4	5	5
N_3	$G_{2,2}$	$\mathbb{R}(4)$	C_2	$[1,1,0]$	$[1,19,12]$	1	16	17	19
N_5	$G_{3,3}$	$\mathbb{R}(8)$	C_2	$[1,1,0]$	$[1,71,56]$	1	64	65	71
N_7	$G_{4,4}$	$\mathbb{R}(16)$	C_2	$[1,1,0]$	$[1,271,240]$	1	256	257	271

Table 5. Classes Ω_{2k-1} for $1 \leq k \leq 4$ and $p - q = 1 \pmod 8$

Class	Group	$C\ell_{p,q}$	Center	C.O.S.	G.O.S.	L	M	N	t
Ω_1	$G_{2,1}$	${}^2\mathbb{R}(2)$	$C_2 \times C_2$	$[1,3,0]$	$[1,11,4]$	2	8	10	11
Ω_3	$G_{3,2}$	${}^2\mathbb{R}(4)$	$C_2 \times C_2$	$[1,3,0]$	$[1,39,24]$	2	32	34	39
Ω_5	$G_{4,3}$	${}^2\mathbb{R}(8)$	$C_2 \times C_2$	$[1,3,0]$	$[1,143,112]$	2	128	130	143
Ω_7	$G_{5,4}$	${}^2\mathbb{R}(16)$	$C_2 \times C_2$	$[1,3,0]$	$[1,543,480]$	2	512	514	543

Theorem 4. *Each irreducible character of $G_{p,q}$ in classes N_{2k-1} and Ω_{2k-1} for $1 \leq k \leq 4$ can be realized over $\mathbb{R}$.*

[9] The Radon-Hurwitz number r_i is defined by recursion as $r_{i+8} = r_i + 4$ and these initial values: $r_0 = 0, r_1 = 1, r_2 = r_3 = 2, r_4 = r_5 = r_6 = r_7 = 3$.

Proof. The proof is based on the Frobenius-Schur Count of Involutions (29) applied to each class group $G_{p,q}$ listed in Tables 4 and 5, and showing that for every irreducible character χ of $G_{p,q}$, its indicator function $\iota\chi = 1$. That is, the sum

$$\sum_{i=1}^{N} (\iota\chi_i)\chi_i(1) = 1 + t \tag{32}$$

is maximum in the sense that each coefficient $\iota\chi_i$ of the character degree $\chi_i(1)$ equals 1.

For the groups in classes N_{2k-1}, we have only one faithful nonlinear character χ_N which needs to be shown that it can be realized over $\mathbb{R}$. The remaining M characters χ_i for $1 \leq i \leq M$ $(M = N - 1)$ are linear and real, hence they can be realized over $\mathbb{R}$. Furthermore, we have

$$|G_{p,q}| = 2^{1+p+q} = M + \sum_{i=M+1}^{N} m_i^2 \tag{33}$$

where $m_i = \dim V^{(i)}$ is the dimension of an irreducible $\mathbb{C}G$-submodule of the regular module $\mathbb{C}G$ in a decomposition $\mathbb{C}G = \oplus_i m_i V^{(i)}$.[10]

For the two groups in the class N_1, we have already established this result in Example 1. For the two groups $G_{2,2} \cong G_{3,1}$ in N_3, we have

$$\sum_{i=1}^{M=16} (\iota\chi_i)\chi_i(1) + (\iota\chi_{17})\chi_{17}(1) = 1 + t = 20, \tag{34a}$$

$$2^{1+p+q} = 32 = 16 + m_{17}^2 \tag{34b}$$

which gives $m_{17} = \chi_{17}(1) = 4$ and $\iota\chi_i = 1$ for $1 \leq i \leq 17$. For the three groups $G_{0,6} \cong G_{3,3} \cong G_{4,2}$ in the class N_5, we find that $m_{65} = \chi_{65}(1) = 8$ and $\iota\chi_i = 1$ for $1 \leq i \leq 65$. Finally, for the five groups $G_{0,8} \cong G_{1,7} \cong G_{4,4} \cong G_{5,3} \cong G_{8,0}$ we find that $m_{257} = \chi_{257}(1) = 16$ and again $\iota\chi_i = 1$ for $1 \leq i \leq 257$.

For the single group $G_{2,1}$ in the class Ω_1, we have

$$\sum_{i=1}^{M=8} (\iota\chi_i)\chi_i(1) + (\iota\chi_9)\chi_9(1) + (\iota\chi_{10})\chi_{10}(1) = 1 + t = 12, \tag{35a}$$

$$2^{1+p+q} = 16 = 8 + m_9^2 + m_{10}^2 \tag{35b}$$

which gives $m_9 = \chi_9(1) = m_{10} = \chi_{10}(1) = 2$ and $\iota\chi_i = 1$ for $1 \leq i \leq 10$. Likewise, for the single group $G_{3,2}$ in the class Ω_3, we find that $m_{33} = \chi_{33}(1) = m_{34} = \chi_{34}(1) = 4$ and $\iota\chi_i = 1$ for $1 \leq i \leq 34$. For the two groups $G_{0,7} \cong G_{4,3}$ in the class Ω_5, we have $m_{129} = \chi_{129}(1) = m_{130} = \chi_{130}(1) = 8$ and $\iota\chi_i = 1$ for $1 \leq i \leq 130$. Finally, for the three groups $G_{1,8} \cong G_{5,4} \cong G_{9,0}$ in the class Ω_7, we deduce that $m_{513} = \chi_{513}(1) = m_{514} = \chi_{514}(1) = 16$ and $\iota\chi_i = 1$ for $1 \leq i \leq 514$. □

[10] In general, these modules are complex. In our case, these modules are real since all irreducible characters can be realized over $\mathbb{R}$.

Theorem 5. *Suppose we have Clifford algebra*

$$C\ell_{p,q} \cong \begin{cases} \mathbb{R}(2^l), & p-q = 0,2 \pmod 8, \\ {}^2\mathbb{R}(2^{l-1}), & p-q = 1 \pmod 8, \end{cases}$$

for $p+q \le 9$, where $l = q - r_{q-p}$ equals the number of factors in a primitive idempotent f defining $S = C\ell_{p,q}f$, a minimal spinor ideal. Let $G_{p,q}$ be the Salingaros vee group contained in $C\ell^*_{p,q}$. Let $1 \le k \le 4$. Then,

(i) *$G_{p,q}$ is in the class N_{2k-1} or Ω_{2k-1}, and all irreducible characters of $G_{p,q}$ can be realized over $\mathbb{R}$.*

(ii) *The degree of a nonlinear character χ of $G_{p,q}$ equals the dimension of the real spinor space $S = C\ell_{p,q}f$. In particular,*

$$\chi(1) = \dim_{\mathbb{R}} S = \begin{cases} 2^l, & \text{when } p-q = 0,2 \pmod 8, \\ 2^{l-1} & \text{when } p-q = 1 \pmod 8, \end{cases} \tag{36}$$

where $l = q - r_{q-p}$ and r_i is the Radon-Hurwitz number. Thus,

$$q - r_{q-p} = l = \begin{cases} \log_2 \chi(1), & \text{when } p-q = 0,2 \pmod 8, \\ 1 + \log_2 \chi(1) & \text{when } p-q = 1 \pmod 8. \end{cases} \tag{37}$$

(iii) *Every nonlinear character of $G_{p,q}$ uniquely determines a spinor module of $C\ell_{p,q}$. When $G_{p,q}$ is in the class N_{2k-1} (resp. Ω_{2k-1}), the spinor representation is faithful (resp. non faithful).*

Proof of (i) and (ii): See Thm. 4 and its proof.

Proof of (iii) (sketch): We know from Thm. 2 that spinor modules of $C\ell_{1,1}$ and $C\ell_{2,0}$ are uniquely determined by the nonlinear character of $D_8 \cong G_{1,1} \cong G_{2,0}$ of degree 2. In general, we know from [1] that $C\ell_{p,q} \cong \mathbb{R}[G_{p,q}]/\mathcal{J}$. Furthermore, any irreducible nonlinear character χ of $G_{p,q}$ for $p-q = 0,1,2 \pmod 8$ can be realized over $\mathbb{R}$ by Thm. 4 so that $\chi(1) = \dim_{\mathbb{R}} S$ by (ii). Thus, the spinor module of $C\ell_{p,q}$ is determined by a primitive idempotent which is the image of a primitive idempotent determined by χ.

When $G_{p,q}$ is in the class N_{2k-1}, the unique nonlinear character is faithful, hence, the spinor representation is faithful. For $k=1$, see the character table for D_8 (Table 1) since D_8 is in the class N_1. For groups in the classes N_3, N_5, and N_7, we give abbreviated character tables (Tables 6, 7, and 8) showing the only nonlinear character in each class.

TABLE 6. Nonlinear character of $G_{p,q}$ in class N_3

C	$1a$	$2a$	$2b \longleftrightarrow 4f$
$\lvert C \rvert$	1	1	$2 \longleftrightarrow 2$
χ_{17}	4	-4	$0 \longleftrightarrow 0$

TABLE 7. Nonlinear character of $G_{p,q}$ in class N_5

C	$1a$	$2a$	$2b \longleftrightarrow 4ab$
$\|C\|$	1	1	$2 \longleftrightarrow 2$
χ_{65}	8	-8	$0 \longleftrightarrow 0$

TABLE 8. Nonlinear character of $G_{p,q}$ in class N_7

C	$1a$	$2a$	$2b \longleftrightarrow 4dp$
$\|C\|$	1	1	$2 \longleftrightarrow 2$
χ_{257}	16	-16	$0 \longleftrightarrow 0$

When $G_{p,q}$ is in the class Ω_{2k-1}, there are two different nonlinear non faithful characters, each of which determines a non faithful spinor representation of $C\ell_{p,q}$. When $k = 1$, see Table 12 of $G_{2,1}$ of class Ω_1. For groups in the classes Ω_3, Ω_5, and Ω_7, see the abbreviated character tables (Tables 9, 10, and 11) displaying the only two nonlinear characters in each class.

TABLE 9. Nonlinear characters of $G_{p,q}$ in class Ω_3

C	$1a$	$2a$	$2b$	$2c$	$2d \longleftrightarrow 4l$
$\|C\|$	1	1	1	1	$2 \longleftrightarrow 2$
χ_{33}	4	-4	-4	4	$0 \longleftrightarrow 0$
χ_{34}	4	-4	4	-4	$0 \longleftrightarrow 0$

TABLE 10. Nonlinear characters of $G_{p,q}$ in class Ω_5

C	$1a$	$2a$	$2b$	$2c$	$2d \longleftrightarrow 4bd$
$\|C\|$	1	1	1	1	$2 \longleftrightarrow 2$
χ_{129}	8	-8	8	-8	$0 \longleftrightarrow 0$
χ_{130}	8	8	8	-8	$0 \longleftrightarrow 0$

Complete character tables for groups N_{2k-1} and Ω_{2k-1} for $k = 2, 3, 4$ were derived with `Maple` [10]. In Tables 6, 7, and 8, the first two conjugacy classes C are central and are labeled by $1a$ and $2a$ whereas all remaining classes are labeled by $2b \longleftrightarrow 4f$, $2b \longleftrightarrow 4ab$, and $2b \longleftrightarrow 4dp$, respectively. Each label contains an integer, which gives the order of each element in the class, whereas the letters give a consecutive number (in the base 26) of the class with elements of that order. The number of elements in the class C is given by $|C|$. It is easily seen that the single nonlinear character is faithful. Similarly in Tables 9, 10, and 11, except that the first four conjugacy classes C are central, and each group has two distinct nonlinear irreducible characters.

TABLE 11. Nonlinear characters of $G_{p,q}$ in class Ω_7

C	$1a$	$2a$	$2b$	$2c$	$2d \longleftrightarrow 4if$
$\|C\|$	1	1	1	1	$2 \longleftrightarrow 2$
χ_{513}	16	−16	−16	16	$0 \longleftrightarrow 0$
χ_{514}	16	16	−16	16	$0 \longleftrightarrow 0$

It is easily seen that these characters are non faithful. Finally, one can observe that the degrees of the nonlinear characters of all these groups were correctly computed in the proof of Thm. 4. □

We illustrate Thm. 5 on $\mathbb{R}[G_{2,1}]/\mathcal{J} \cong C\ell_{2,1}$ for $G_{2,1}$ in the class Ω_1.

Example 3. *Let*

$$G = \langle \tau, g_1, g_2, g_3 \mid \tau^2 = g_1^2 = g_2^2 = 2, g_3^4 = 1, \tau g_i = g_i \tau, \\ g_i g_j = \tau g_j g_i, i, j = 1, 2, 3 \rangle. \quad (38)$$

Then, $G \cong G_{2,1}$, $Z(G) = \{1, \tau, g_1g_2g_3, \tau g_1g_2g_3\} \cong C_2 \times C_2$, *the group order structure is* $[1, 11, 4]$, *and* G *has ten conjugacy classes:*

$$K_1 = \{1\}, \quad K_2 = \{\tau\}, \quad K_3 = \{g_1g_2g_3\}, \quad K_4 = \{\tau g_1g_2g_3\}, \\ K_5 = \{g_1, \tau g_1\}, \quad K_6 = \{g_2, \tau g_2\}, \quad K_7 = \{g_3, \tau g_3\}, \\ K_8 = \{g_1g_2, \tau g_1g_2\}, \quad K_9 = \{g_1g_3, \tau g_1g_3\}, \quad K_{10} = \{g_2g_3, \tau g_3g_3\}. \quad (39)$$

A character table of G *is given in Table* 12. *Let* $\theta_1 = \frac{1}{2}(1+\tau)$ *and* $\theta_2 = \frac{1}{2}(1-\tau)$ *be two central orthogonal idempotents in the group algebra* $\mathbb{R}[G]$ *adding to* 1. *All characters of* G *can be realized over* $\mathbb{R}$ *because the trivial*

TABLE 12. Character table for $G_{2,1}$ in class Ω_1

class	K_1	K_2	K_3	K_4	K_5	K_6	K_7	K_8	K_9	K_{10}
$\|C_G(g_i)\|$	16	16	16	16	8	8	8	8	8	8
χ_1	1	1	1	1	1	1	1	1	1	1
χ_2	1	1	−1	−1	1	1	−1	1	−1	−1
χ_3	1	1	−1	−1	1	−1	1	−1	1	−1
χ_4	1	1	1	1	1	−1	−1	−1	−1	1
χ_5	1	1	−1	−1	−1	1	1	−1	−1	1
χ_6	1	1	1	1	−1	1	−1	−1	1	−1
χ_7	1	1	1	1	−1	−1	1	1	−1	−1
χ_8	1	1	−1	−1	−1	−1	−1	1	1	1
χ_9	2	−2	−2	2	0	0	0	0	0	0
χ_{10}	2	−2	2	−2	0	0	0	0	0	0

character $1_G = \chi_1$ *is a constituent of* χ_S *for each irreducible character* χ *of* G.

The regular $\mathbb{R}G$*-module* $\mathbb{R}G$ *is a direct sum of irreducible* $\mathbb{R}G$*-submodules*

$$\mathbb{R}G = \left(\oplus_{i=1}^{8} U_i\right) \oplus U_9 \oplus U_9' \oplus U_{10} \oplus U_{10}' \tag{40}$$

where the character of U_i *is* χ_i*, when* $1 \leq i \leq 8$*, the character of* $U_9 \oplus U_9'$ *is* $2\chi_9$*, and the character of* $U_{10} \oplus U_{10}'$ *is* $2\chi_{10}$*. Let* $W_1 = \oplus_{i=1}^{8} U_i$*,* $W_2 = U_9 \oplus U_9'$*, and* $W_3 = U_{10} \oplus U_{10}'$*. Then,* W_1, W_2, W_3 *do not have a common composition factor and* $\dim W_1 = 8$, $\dim W_2 = \dim W_3 = 4$*. For each* i $(1 \leq i \leq 8)$*, using formula* (13) *we compute an idempotent* e_i*. Let* e_9 *and* e_{10} *be the idempotents determined by the characters of* W_2 *and* W_3*, respectively. Then,*

$$e_1 + \cdots + e_8 = \frac{1}{2}(1 + \tau) = \theta_1, \tag{41}$$

$$e_9 + e_{10} = \frac{1}{2}(1 - \tau) = \theta_2. \tag{42}$$

It can be easily verified that the idempotents $e_1, \ldots, e_{10}$ *give an orthogonal decomposition of the unity in* $\mathbb{R}[G]$*. Thus,*

$$\mathcal{J} = (1 + \tau) = (\theta_1) = \mathbb{R}[G]\theta_1 = \mathrm{sp}\{e_1, \ldots, e_8\}, \tag{43}$$

$$\mathbb{R}[G]\theta_2 = \mathrm{sp}\{q_1, q_2, \ldots, q_8\} \tag{44}$$

where $q_1 = e_9 + e_{10}$ *and* $q_2 = g_1g_2g_3q_1$, $q_3 = g_1q_1$, $q_4 = g_2q_1, q_5 = g_3q_1$, $q_6 = g_1g_2q_1$, $q_7 = g_1g_3q_1$, $q_8 = g_2g_3q_1$*. Therefore,*

$$\mathbb{R}[G]\theta_2 \cong \mathbb{R}[G]/\mathcal{J} \cong C\ell_{2,1}. \tag{45}$$

Table 13 *shows a multiplication table for the basis elements of* $\mathbb{R}[G]\theta_2$*.*

Table 13. Multiplication table in $\mathbb{R}[G]\theta_2 \cong C\ell_{2,1}$

	q_1	q_2	q_3	q_4	q_5	q_6	q_7	q_8
q_1	q_1	q_2	q_3	q_4	q_5	q_6	q_7	q_8
q_2	q_2	q_1	q_8	$-q_7$	$-q_6$	$-q_5$	$-q_4$	q_3
q_3	q_3	q_8	q_1	q_6	q_7	q_4	q_5	q_2
q_4	q_4	$-q_7$	$-q_6$	q_1	q_8	$-q_3$	$-q_2$	q_5
q_5	q_5	$-q_6$	$-q_7$	$-q_8$	$-q_1$	q_2	q_3	q_4
q_6	q_6	$-q_5$	$-q_4$	q_3	q_2	$-q_1$	$-q_8$	q_7
q_7	q_7	$-q_4$	$-q_5$	$-q_2$	$-q_3$	q_8	q_1	q_6
q_8	q_8	q_3	q_2	$-q_5$	$-q_4$	$-q_7$	$-q_6$	q_1

Thus, q_1 *is the identity of* $\mathbb{R}[G_{2,1}]\theta_2$*, and* $|q_3| = |q_4| = 2$ *while* $|q_5| = 4$*. Furthermore,* $q_3q_4 = -q_4q_3$*,* $q_3q_5 = -q_5q_3$*, and* $q_4q_5 = -q_5q_4$*. Thus, we can define the following* $\mathbb{R}$*-algebra isomorphism on generators:*

$$\phi : \mathbb{R}[G_{2,1}]\theta_2 \to C\ell_{2,1} \quad \textit{with} \quad q_1 \mapsto 1,\ q_3 \mapsto \mathbf{e}_1,\ q_4 \mapsto \mathbf{e}_2,\ q_5 \mapsto \mathbf{e}_3 \tag{46}$$

and then extend it by linearity to all elements in the domain. As expected, $Z(\mathbb{R}[G_{2,1}]\theta_2) = \{q_1, q_2\}$ *and since* $q_2^2 = q_1$*, we can project out two simple* *subalgebras with a help of two central and orthogonal idempotents* $F_1 = \frac{1}{2}(q_1 - q_2)$ *and* $F_2 = \frac{1}{2}(q_1 + q_2)$*:*

$$C\ell_{p,q} \cong \mathbb{R}[G_{2,1}]\theta_2 = \mathbb{R}[G_{2,1}]F_1 \oplus \mathbb{R}[G_{2,1}]F_2 = \mathbb{R}[G_{2,1}]e_9 \oplus \mathbb{R}[G_{2,1}]e_{10} \quad (47)$$

since $F_1 = e_9$ *and* $F_2 = e_{10}$*. Thus, the two four-dimensional simple subalgebras of* $\mathbb{R}[G_{2,1}]\theta_2$ *are uniquely determined by the non-linear irreducible (non-faithful) characters* χ_9 *and* χ_{10}*. It follows that*

$$C\ell_{2,1}J_1 \cong \mathbb{R}[G_{2,1}]e_9 = \mathrm{sp}\{q_1e_9, q_3e_9, q_4e_9, q_5e_9\} \cong \mathbb{R}(2), \quad (48)$$

$$C\ell_{2,1}J_2 \cong \mathbb{R}[G_{2,1}]e_{10} = \mathrm{sp}\{q_2e_{10}, q_8e_{10}, q_4e_{10}, q_5e_{10}\} \cong \mathbb{R}(2). \quad (49)$$

where J_1, J_2 *are two central idempotents in* $C\ell_{2,1}$ *defined as* $\frac{1}{2}(1 - \mathbf{e}_{123})$ *and* $\frac{1}{2}(1 + \mathbf{e}_{123})$*, respectively.*

In order to derive "spinorial" non-equivalent representations of the algebra $\mathbb{R}[G_{2,1}]\theta_2$*, we need to further decompose idempotents* e_9 *and* e_{10} *–which are related to reducible* $\mathbb{R}G$*-submodules* W_2 *and* W_3 *of dimension four each– into sums of primitive idempotents. Define*

$$\mathbf{f}_1 = \frac{1}{2}(q_1 + q_3)e_9 \quad \textit{and} \quad \mathbf{f}_1' = \frac{1}{2}(q_1 - q_3)e_9, \quad (50)$$

$$\mathbf{f}_2 = \frac{1}{2}(q_2 + q_8)e_{10} \quad \textit{and} \quad \mathbf{f}_2' = \frac{1}{2}(q_2 - q_8)e_{10}, \quad (51)$$

so $e_9 = \mathbf{f}_1 + \mathbf{f}_1'$, $\mathbf{f}_1^2 = \mathbf{f}_1$, $\mathbf{f}_1'^2 = \mathbf{f}_1'$, $\mathbf{f}_1\mathbf{f}_1' = \mathbf{f}_1'\mathbf{f}_1 = 0$*, and* $e_{10} = \mathbf{f}_2 + \mathbf{f}_2'$, $\mathbf{f}_2^2 = \mathbf{f}_2$, $\mathbf{f}_2'^2 = \mathbf{f}_2'$, $\mathbf{f}_2\mathbf{f}_2' = \mathbf{f}_2'\mathbf{f}_2 = 0$*. Clearly, these four idempotents are primitive as they correspond to four irreducible* $\mathbb{R}G$*-submodules* U_9 *and* U_9'*, and* U_{10} *and* U_{10}'*, respectively, from* (40)*. The two spinorial non faithful but inequivalent representations of* $\mathbb{R}[G_{2,1}]\theta_2$ *can now be realized in say* U_9 *and* U_{10} *with the character* χ_9 *and* χ_{10}*, respectively. We define left ideals:*

$$S_1 = \mathbb{R}[G_{2,1}]\theta_2 e_9\mathbf{f}_1 = \mathrm{sp}\{\mathbf{f}_1, q_4\mathbf{f}_1\}, \quad S_2 = \mathbb{R}[G_{2,1}]\theta_2 e_{10}\mathbf{f}_2 = \mathrm{sp}\{\mathbf{f}_2, q_4\mathbf{f}_2\}. \quad (52)$$

It can be verified that we obtain two (left) representations

$$\rho_9 : \mathbb{R}[G_{2,1}]\theta_2 \to \mathrm{End}(S_1), \quad \rho_{10} : \mathbb{R}[G_{2,1}]\theta_2 \to \mathrm{End}(S_2), \quad (53)$$

namely,

$$\rho_9 : \quad q_3 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad q_4 \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad q_5 \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad (54)$$

$$\rho_{10} : \quad q_3 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad q_4 \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad q_5 \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \quad (55)$$

The two representations ρ_9 *and* ρ_{10} *are obviously non-equivalent yet irreducible as their characters* χ_9 *and* χ_{10} *are irreducible and distinct. Furthermore,* ρ_9 *and* ρ_{10} *are non faithful since their characters are non faithful:* $\ker\chi_9 = \{1, \tau g_1g_2g_3\}$ *and* $\ker\chi_{10} = \{1, g_1g_2g_3\}$*. Thus, as is the case in semisimple Clifford algebras, a faithful representation of* $C\ell_{2,1}$ *is realized in the direct sum* $S_1 \oplus S_2$ *referred to as a double spinor space [8].*

With Tables 6–11, more examples can be constructed to illustrate Thm. 5. The actual knowledge of complete character tables is not needed since the idempotents related to the linear characters span the ideal $\mathcal{J}$, and only the nonlinear characters shown in these tables give spinor representations of the related Clifford algebras.

5. Conclusions

Our two main results stated as Thm. 4 and 5 have been proven for Salingaros vee groups $G_{p,q}$ in classes N_{2k-1} and Ω_{2k-1} for $1 \leq k \leq 4$. We conjecture that they extend to the vee groups in all classes N_{2k-1} and Ω_{2k-1} for $k \geq 1$, that is, to all Clifford algebras $C\ell_{p,q}$ for $p - q = 0, 1, 2 \,(\mathrm{mod}\, 8)$.

Our next step is to apply these ideas to the remaining three classes of Clifford algebras $C\ell_{p,q}$ when $p - q = 4, 6 \,(\mathrm{mod}\, 8)$ (resp. $p - q = 5 \,(\mathrm{mod}\, 8)$, and $p - q = 3, 7 \,(\mathrm{mod}\, 8)$), which are related to the Salingaros classes N_{2k} (resp. Ω_{2k}, and S_k). Since any group in the class N_{2k} is of the form $(D_8)^{\circ(k-1)} \circ Q_8$, while any group in the class Ω_{2k} is the central product $((D_8)^{\circ(k-1)} \circ Q_8) \circ (C_2 \times C_2)$ [1], that is, each of them contains Q_8 as a normal subgroup, we expect to see that quaternionic and double-quaternionic spinor representations emerge, as expected, when $p - q = 4, 5, 6 \,(\mathrm{mod}\ 8)$. The vee groups in the classes S_k are isomorphic to $(D_8)^{\circ k} \circ C_4 \cong (D_8)^{\circ(k-1)} \circ Q_8 \circ C_4$ and we expect that their nonlinear characters yield complex spinor representations. Thus, in all classes we expect to relate the spinor modules of Clifford algebras to the irreducible nonlinear characters of their corresponding vee groups.

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